Analytical Stress Functions applied to Hydraulic Fracturing: Scaling the Interaction of Tectonic Stress and Unbalanced Borehole Pressures

Weijermars, R.

Department of Geotechnology, Delft University of Technology, Stevinweg 1, Delft 2628CN, the Netherlands.

ABSTRACT: Some 19,000 new oil and gas wells were drilled in the US alone in 2010, and many need hydraulic fracturing aimed at improving well productivity. The Kirsch equations have been widely applied to model borehole stresses; in the basic formulation these equations are valid only for isotropic elastic media with a non-pressurized hole. A stress function, from which the Kirsch equations can be derived, is developed here to account for the effect of hydraulic pressure on the cylindrical surface of the wellbore. The concise solution via the stress function superposition is simpler than any previous analytical derivation. The function can be scaled to map the typical stress patterns for unbalanced hydraulic well bore pressures and provides a more comprehensive solution than the early Kirsch equations. The stress fields visualized can account for both over-pressurized and under-pressurized boreholes - with and without a tectonic background stress - and are valid for horizontal and vertical borehole sections. These analytical solutions allow fast mapping of stress trajectories around wellbores, which is useful for applications that aim to optimize hydraulic fracturing and thereby improve oil and gas well productivity. The analytical insight developed here may also contribute to develop better visual images (using stress trajectories) to prevent blow-outs and loss of drilling fluid in damaged formations.

1. INTRODUCTION

The expressions commonly used for elastic stress in studies to predict borehole breakouts and hydraulic fracture patterns for well stimulation have been developed by Ernst Gustav Kirsch, a 19th Century German Engineer, and published over a century ago. The Kirsch [1] formulas of 1898 properly account for the stresses around so-called balanced boreholes, where the drilling fluid supports no deviatoric stress but provides a median pressure on the well bore that equals the lithostatic pressure. When the borehole is pressurized either more or less than the ambient confining pressure in the host rock, the effect of over-pressure or under-pressure on the stress pattern induced in the vicinity of the wellbore can be profound.

A set of analytical equations derived here from the superposition of a comprehensive pair of stress functions accurately models the interaction of hydraulically induced pressures on a borehole with and without a regional tectonic background stress. Previous work on borehole breakout has accounted for unbalanced pressures in boreholes [2, 3]. These earlier solutions incorporate an elasto-plastic solution for shear stress concentrations that determine the locations of actual plastic slip. In this study, a different approach is taken, starting from simple stress function superposition to arrive at an elastic solution that accounts for the stress trajectories around pressurized wellbores.

The approach taken here seeks analytical simplicity by adopting the following model assumptions:

(i) Elastic rock behavior is assumed. Elasto-plastic extensions are possible [2, 3], but not necessary for the initiation of elastic failure and pressure opening of pre-existing surfaces of failure, the application focus of this study.

(ii) Unlike the assumption in previous studies that require communication between pore fluid in the host rock and the drilling fluid (e.g., [4]), such a connection is not essential in the approach outlined here. The drilling fluid in this study simply exerts a static pressure on the inner wellbore surface with an elastic response of the wall rock. In fact, the applicability of elasto-plastic solutions to the impervious rock formations that host unconventional gas (tight sandstone,
gas shales) is likely to be invalid – no pore connectivity exists in such reservoirs.

(iii) By aligning principal stresses and wellbore orientation (both being either vertical or horizontal, Fig. 1), the additional notational complexity arising from the full 3D tensor solution is deliberately avoided, for clarity of physical reasoning. This way the analysis can focus on the essential interaction of the wellbore pressure versus ambient stress system. Boreholes oblique to the principal stress would require expansion with tensor notation, but directional drilling is now nearly abandoned in the oil & gas business [5]. As borehole sections for oil & gas production are now commonly completed either vertical or horizontal, tectonic stresses align with a majority of modern boreholes.

(iv) Stress functions provide appropriate descriptions of the stresses developing in the elastic rock near the wellbore. Stress functions are useful for application to elastic materials if it can be assumed that the stress field is time-independent until the moment of incipient failure. Rheological studies have revealed that rocks at shallow crustal depths fail brittle after the elastic deformation limit is reached, while ductile creep is favored at deeper crustal levels [6]. Modern laboratory data on rock mechanics suggest that the criterion for frictional slip is largely independent of the rock type [6, 7]. The brittle ductile transition occurs at 5 to 8 km depth dependent of the local geothermal gradient. Consequently, most boreholes do not enter the brittle ductile transition and ductile flow can be ruled out, except when rock salt is penetrated, which flows ductile with a low yield stress and at very shallow depths [8]. For rocks deforming at depths that allow ductile creep, the stress field is continually changing dynamically, and stream functions provide a more appropriate description [9] – but there is a strong analogy between stress functions and stream functions links these analytical solutions.

(v) In hydro-fracturing, pre-existing fractures and cleats dilate due to the applied pressure (no need for plastic yield criterion) and the equations developed show the pressure required to separate the fracture walls when borehole pressures exceed the deviatoric stress that keep the rock fractures closed before the hydraulic pressure is applied. Arguably, most of frac jobs (unlike the name suggestion) are in fact opening of existing freeways rather than the creation of new fractures. The fact that seismicity traces frac fluid propagation as far as 1,500 feet from the borehole can be explained by the opening of pre-existing failure surfaces (see later).

2. PRACTICAL IMPORTANCE

A suitable set of stress functions is here superposed to model the fractures around boreholes accounting for the interaction between the hydraulic wellbore pressure and a regional background stress. The recovery of natural gas from unconventional resources frequently requires well completion involving hydraulic fracturing to enhance well productivity. Fracturing fairways are opened in frac pumping cycles to enhance the recovery of gas in place. Unconventional gas development typically uses a combination of 40 acre spacing vertical wells, 20 acre spacing vertical wells, and 80 acre spacing horizontal wells. Slim hole technology uses 5 3/4 inch (or even micro-holes diameters ranging from 1.25 to 2.38 inches), instead of 8 3/4 inch conventional completion size [10].

The oil and gas industry’s capital expenditure on well stimulation is considerable, with typical cost for well completion built up as follows: 20-25% drilling rig, 30-35% frac job, 10-15% tubulars [11]. Optimization of production therefore critically depends on the success of the enhanced reservoir connectivity induced by so-called frac jobs. The hydraulic fluid pumped into the wellbore is pressurized to induce fractures along predetermined slots. The slots themselves are engineered to align with the direction of tectonic stress field such that maximum rupture occurs within the production layer. The hydraulic fracturing job uses fluids (mostly water, sometimes acidized to dissolve carbonate matrix) and proppants (mostly sand), with nitrogen-foamed fracturing fluid being common for shallow shales with low reservoir pressure. The pumped fluid reaches pressure of 8,000 psi (55 MPa) [12]. Micro-seismic recordings at an observation well, during hydraulic frac jobs in an
adjacent vertical well, have revealed that fractures propagate (or pre-existing fissures dilate) in shale formations [13] and tight sands [14] as much as 1,500 feet (~0.5 km) in lateral directions.

Brittle strength tests on unfractured rock samples have shown that the preferred mode of failure beyond the elastic limit includes tension joints and shear joints (Fig. 2). Although, tension joints and shear joints may form simultaneously, shear joints are favoured over tension joints at higher confining pressures, while tension joints prevail at low confining pressures of near surface depths (e.g., [15]). This knowledge of the orientation of failure planes in rocks (Fig. 2) can be applied to predict failure surfaces in nature, provided the state of stress in the regional rock volume is known.

Shear joints form close to the theoretical direction of maximum shear stress, which is in two conjugate planes at 45° to \( \sigma_1 \) and \( \sigma_3 \); the bisector of the conjugate shear planes is parallel to \( \sigma_2 \). It is well known that the actual angle of shear failure (angle of internal friction, \( \phi \)) may differ from the theoretical 45° to \( \sigma_1 \) (\( \phi \) commonly is at 30° to 40° to \( \sigma_1 \)), but this disparity between empirical shear failure planes and theoretical slip surfaces is neglected in what follows. This does not limit, but amplifies the application range of the present results, because for each particular rock unit a definite, determinable angular offset exists between theoretical slip lines and the empirical surfaces of shear failure. If the angle of internal friction of the material in which slip lines occur is zero, then deviatoric stresses - if locally exceeding the yield stress - will cause shear failure along the direction of the theoretical slip surfaces calculated in this study.

### 3. STRESS FUNCTIONS AND STRESS TRAJECTORIES

Stress functions are useful analytical tools which can be utilized to predict the orientation of principal stresses and failure surfaces in elastic solids. Analytical methods for predicting elastic failure have a long history in mechanical engineering applications [16], where stress functions are used to predict stress concentrations and frictional plasticity on slip surfaces or ‘shear zones’ in manufacturing tools. Elastic failure in rocks under natural deviatoric stresses has been successfully modeled by predicting thrust fault orientation from a range of stress functions [17, 18]. Individual stress trajectories can be mapped using specific stress functions, each of which applies to a particular stress field only.

Stress functions have been previously applied in structural geology studies to predict the orientation of thrust faults [17-21] and volcanic dike injection patterns [22-24]. The slip-line field theory, which is based on analytical stress functions, has also been successfully applied to the study of mega-shears resulting from the collision of the Indian and Eurasian plates [25-27], and to stresses arising around pressurized domes [28, 29]. Stress functions (and related complex potentials) have also provided analytical descriptions for tilt and displacement gradients around 2D elliptical model cracks [30-33] and 3D model cracks [34, 35].

The spatial relationships of the forces on particles of a continuous isotropic elastic medium are subject to the requirements of equilibrium and compatibility of stress. These two conditions must be accounted for by any valid stress function (Appendices A and B), which then describes both the spatial orientation of stress field and the

**Fig. 2.** Tension joints in laboratory tests are formed perpendicular to the least principal total stress axis, \( \sigma_3 \), and parallel to the \( \sigma_1-\sigma_2 \) plane. Orientation of: (a) tension joints, and (b) shear joints with respect to the total principal stresses.

**Fig. 3(a).** Uniform stress field with constant orientation of principal stresses can be outlined by orthogonal set of stress trajectories. The magnitude of the stresses may or may not vary spatially. (b) Non-uniform stress field where the orientation of the principal stress axes vary in space. Implicitly, the magnitude of the principal stresses will then also vary spatially, commonly with steeper gradients where the curvature of the trajectories changes fastest.
stress magnitudes inside each point of a mechanical continuum. Variations in the spatial orientation of the principal stresses can be effectively illustrated by stress trajectories, mapped from the stress function.

Stress trajectories display graphically the spatial variation in the orientation of the principal stress axes, which is extremely useful in geological and petroleum engineering applications. The state of stress in an infinitely small point within the stressed rock can be graphically represented by a stress ellipsoid, which fixes the relative magnitude of the three axes of the principal stress. If the three principal axes of the stress ellipsoid inside a material are similarly oriented in each point of the material, the stress field is uniform. The stress trajectories for a uniform stress field are comprised of an orthogonal set of straight gridlines (Fig. 3a). The orientation of the stress axes inside an isotropic elastic medium does not necessarily need to maintain a uniform orientation throughout that medium. For example, stress trajectories for a non-uniform stress field delineate the principal stress directions by an orthogonal set of smoothly curved quadrilaterals (Fig. 3b).

The definition of Airy’s stress function \( \Phi \) is a potential field (with units: Pa m) with the Laplacian operator \( \nabla^2 \) satisfied in plane stress problems (in the XZ plane, Fig. 3a). The use of Airy’s stress function is extremely useful in geological and petroleum engineering applications. The state of stress in a material point (for notation conventions see Appendix A) can be described as partial derivatives of \( \Phi \):

\[
(\partial^2 \tau_{x\beta}/\partial x^2)+2\tau_{x\gamma}/\partial x\partial z=2\partial^2 \Phi/\partial x \partial z (2)
\]

More directly, both conditions will be automatically satisfied if the stress function is a valid solution of the bi-harmonic function:

\[
(\partial^4 \Phi/\partial x^4)+(2\partial^4 \Phi/\partial x^2 \partial z^2)+(\partial^4 \Phi/\partial z^4)=0 (3a)
\]

Eq. (3a) can be shortly written as:

\[
\nabla^4 \Phi=0 \quad (3b)
\]

with the Laplacian operator \( \nabla^2 \), \( \nabla^2=\partial^2/\partial x^2+\partial^2/\partial z^2 \), comprised in Eq. (3b) by \( \nabla^4=\nabla^2 \nabla^2 \). The use of Airy’s stress function has been demonstrated in several analytical studies of geological fault patterns [17-19, 21, 22, 28].

Before applying stress functions to the specific example of boreholes, it is useful to prove that stress functions may simply be summed to find the total state of deviatoric stress. The starting equations are:

\[
\begin{align*}
\tau_{xx} & = \tau_{xx1}+\tau_{xx2} \quad (4a) \\
\tau_{yy} & = \tau_{yy1}+\tau_{yy2} \quad (4b)
\end{align*}
\]

Using \( \tau_{zx}=-\partial^2 \Phi/\partial x\partial z \) and \( \tau_{zy}=-\partial^2 \Phi/\partial z \partial x \) together with Eqs. (4a,b) yield:

\[
\begin{align*}
\partial^2 \Phi/\partial x^2 - \partial^2 \Phi/\partial z^2 = \partial^2 \Phi/\partial x \partial z (5a) \\
\partial^2 \Phi/\partial x^2 + \partial^2 \Phi/\partial z^2 = \partial^2 \Phi/\partial x \partial z (5b)
\end{align*}
\]

Similarly, \( \nabla^4 \Phi=\nabla^4 (\Phi_1+\Phi_2) \). It follows that:

\[
\Phi_{\text{total}}=\Phi_1+\Phi_2 \quad (6)
\]

This relationship is of fundamental importance for finding the solution of the stress field resulting from superposing several analytical stress field solutions. The annotation employed here takes \( \sigma \) for total stresses and \( \tau \) for deviatoric stresses, using subscripts to distinguish normal stresses and shear stresses. This usage is in accordance with Fung [36], but at variance with some other textbooks where \( \sigma \) is reserved for normal stress and \( \tau \) for shear components. Fung’s [36] notation is preferred here since it helps to distinguish the deviatoric normal stress \( \tau_{xx}=(\sigma_{xx}-\sigma_{zz})/2 \) and the maximum deviatoric shear stress \( \tau_{zx}=(\tau_{xx}-\tau_{yy})/2 \) in plani-form stress/strain situations.

4. PRINCIPAL STRESS MAGNITUDE AND ORIENTATION

The magnitude of the elements of the deviatoric stress tensor in each material point (for notation conventions see Weijermars [15], table 10-1) can be described as partial derivatives of \( \Phi \):

\[
\begin{align*}
\tau_{zz} & = \partial^2 \Phi/\partial x^2 \quad (7a) \\
\tau_{xz} & = \partial^2 \Phi/\partial z^2 \quad (7b) \\
\tau_{zy} & = \partial^2 \Phi/\partial x \partial z \quad (7c)
\end{align*}
\]

Stress trajectories outline the spatial orientation of the minimum, intermediate, and maximum principal deviatoric stresses; an associated quantity is the direction of maximum shear stress. The lines (planes in 3-D) or curvi-linear surfaces (surfaces in 3-D) that connect the tangents to the directions of the maximum shear stress are termed slip curves (slip surfaces in 3-D).

Slip curves are tangential to the direction of maximum shear stress found at \( +\pi/4 \) to the stress trajectories, so that there are two orthogonal sets of slip curves bisecting the angles between the stress trajectories at 45°. Slip curves making an angle of \( -\pi/4 \) with the \( \tau_{zx} \)-direction are termed \( \alpha \)-curves, and those at \( -\pi/4 \) are called \( \beta \)-curves (Fig. 4a). Slip curves are useful because they outline the trace of surfaces...
of potential shear failure (relevant for wellbore breakouts). No shear failure is likely when $\alpha$-curves intersect at some point A (Fig. 4b). The curvature of $\beta$-curves becomes infinite near A and approaches a circle. Such singular points only possess neutral stress or pressure (deviatoric stresses vanish) and therefore are termed neutral points. The concept of slip curves has been successfully applied to explain the fault pattern associated with indentation of mainland Asia by the Indian subcontinent [25-27]. The angular relationships of the slip curves are subject to Hencky’s Theorems (Appendix C). Upon fracture initiation the elastic stress field will be perturbed around the propagating fracture as modeled elsewhere [37, 38].

5. STRESS FUNCTION FOR REGIONAL TECTONIC STRESS WITHOUT PRESENCE OF BOREHOLE

A uniaxial regional compression (i.e., $\tau_1$ is horizontal and $\tau_3$ is vertical, normal to the ground surface, while $\tau_2=0$) can be represented by a stress function for a uniaxial deviatoric compression within an isotropic elastic plate without a hole (Fig. 5):

$$\Phi_1(x,y)=\left(\frac{\tau_1}{2}\right)y^2$$

(8a)

which expressed in cylindrical coordinates corresponds to:

$$\Phi_1(r,\theta)=\left(\frac{\tau_1}{2}\right)r^2\sin^2\theta$$

(8b)

This translates to geological situations where deviatoric tectonic stresses may reach up to 100 to 1,000 MPa [39, 40], where the regional uni-axial compression generates a deviatoric stress $\tau_{xx} = \tau_1$ and total stress $\sigma = 2\tau_1$.

6. STRESS FUNCTION FOR A PRESSURIZED BOREHOLE WITHOUT A REGIONAL STRESS

The state of stress around a circular borehole solely subject to pressure from a wellbore fluid (mud, or hydraulic fluid in a frac job) is axially symmetric and characterized by a stress trajectory pattern of radial lines and concentric circles defining a spider-web pattern (Fig. 6). This stress field can be described by a simple stress function in a 2-D cylindrical or polar reference frame $(r,\theta)$ (after Timoshenko and Goodier [16]):

$$\Phi_2(r,\theta) = A \ln r$$

(9a)

The value of the parameter $A$ can be expressed as a function of the hydraulic pressure, $P_m$, and the radius of the drill hole, ‘a’. Because the radial stress equals the hydraulic pressure, $P_m$, at $r=a$ it can be shown that $A=P_m a^2$ in order to satisfy this boundary condition. Consequently, Eq. (9a) can be rewritten as:

$$\Phi_2(r,\theta) = P_m a^2 \ln r$$

(9b)

The two normal and symmetric shear elements of the stress tensor are obtained by differentiation according to Eqs. (1a to c), which in cylindrical coordinates transform

Fig. 4(a). Principal directions of stress (solid) and theoretical slip-curves (dashed). (b) Family of $\alpha$-curves meeting at a singular point A.

Fig. 5: Stress trajectories for a virtual vertical or horizontal hole in an elastic slab subject to a uniform regional field stress perpendicular to the axis of the hole.

Fig. 6: Stress trajectories around an internally pressurized circular borehole.
continues to occur even when subsequent load pressures
hydraulic fracturing [41] shows that micro-seismicity
well stimulation by stepped hydraulic fracturing. The
when subjected to 20 to 30 load cycles commonly used in
fissure opening. Only yield strengths of the order of 0.01
yield strengths of 0.1 MPa accounts for 5.21 m laterally
will dilate a tension fissure of 1.65 m lateral length and
example, existing cracks with yield strength of only 1 MPa
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existing fractures are opened by the frac fluid with tensile
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levels.

Hydraulic fracture propagation has been monitored with
micro-seismics to create lateral fracture movement up to
1,500 feet (~0.5 km) from the borehole. This means
existing fractures are opened by the frac fluid with tensile
yield strength much lower than 10 to 15 MPa. For
example, existing cracks with yield strength of only 1 MPa
will dilate a tension fissure of 1.65 m lateral length and
yield strengths of 0.1 MPa accounts for 5.21 m lateral fissure opening. Only yield strengths of the order of 0.01
MPa can account for lateral fissure opening of 16.5 m. The latter would accrue to about 0.5 km dilation movement
when subjected to 20 to 30 load cycles commonly used in
well stimulation by stepped hydraulic fracturing. The
absence of the Kaiser effect during cyclical loading for
hydraulic fracturing [41] shows that micro-seismicity continues to occur even when subsequent load pressures
do not exceed the maximum previously reached load
levels.

7. STRESS FUNCTION FOR A NON-PRESSURIZED
BOREHOLE WITH A REGIONAL STRESS

A regional, tectonic stress field will be deflected by the
emplacement of a borehole (Fig. 7), even if the pressure on
its walls were to be zero, and contribution of Eq. (9a) were
to vanish. The Airy stress function for a circular hole in a
plate subjected to a far field stress corresponding to a
horizontal compression is (Savin [43], p. 106, eq. 2.150):

\[
\Phi_3(r, \theta) = \frac{\tau_\theta}{2} r^2 - \frac{\tau_\theta^2}{4} a^2 \ln(r/a) + \left[ (\tau_\theta/4)r^2 - (\tau/2)a^2 + (\tau/4)(a/r) \right] \cos \theta
\]

This assumes zero litho-static pressure on the walls of the
hole and assumes deviatoric stresses are due to regional
tectonics only. Differentiation of Eq. (13) according to
Eqs. (7a-c) yields stress components similar to the Kirsch
[1] equations (c.f., Timoshenko and Goodier, [16], eqs. 61,
Eqs. (7a-c) yields stress components similar to the Kirsch
[1] equations (c.f., Timoshenko and Goodier, [16], eqs. 61,
P. 91) commonly applied to describe the stress pattern near
boreholes in connection with breakout studies [44,45]. In
an alternative approach, the stress around a free hole
subjected to far field stress has been discussed by Jaeger
and Cook ([42], section 10.4) in terms of a complex
variable function due to Muskhelishvili [46].

Distinguishing between compressive and tensile stresses is
important to correctly represent the stress trajectories
around boreholes as in Fig. 7. The minimum and
intermediate stresses are always horizontal in a normal-
faulting stress state in the crust. In other regions, the
crust may be in a strike-slip faulting stress state for
which the minimum and maximum stresses are
horizontal. In these situations, the minimum stress is
always horizontal and borehole breakout directions

![Fig. 7: Stress trajectories around a cylindrical hole within an uniaxial far field stress. The isotropic points A is where \( \tau_{2}\) = \( \tau_{3}\) = 0, so there is no deviatoric stress in A. (Adapted from Jaeger and Cook [42], fig. 10.4b).](image)
indicate \( \tau \). Stress trajectories are frequently represented 90\(^\circ\) rotated in borehole stress field studies, which occurs when sign reversals are not corrected for in the classical equations. Mechanical engineers traditionally use a positive sign for tensile stress, and geotechnical engineers often take compressive stress as positive.

8. SUPERPOSITION OF TECTONIC STRESS AND PRESSURIZED BOREHOLE

The stress pattern around a pressurized borehole with a regional tectonic background stress can now be concisely expressed by summing the stress functions of Eqs. (9b) and (13) (Figs. 8a & b):

\[
\Phi_\theta(r,0) = \Phi_\phi(r,0)+\Phi_\chi(r,0) = P_m a^2 \ln r + (\tau_\phi/4) r^2 - (\tau_\chi/2) a^2 + (\tau_\phi/4)(a/r)^2 \cos 2 \theta 
\]

(14)

The elements of the stress tensor are obtained by differentiation in cylindrical coordinates, according to Eqs. (10a to c), (valid only for \( r \geq a \)):

\[
\tau_r = P_m(a/r)^2 + (\tau_\phi/2)[1-(a/r)^2]-[1-4(a/r)^2]a^2 + 3(a/r)^4] \cos 2 \theta 
\]

(15a)

\[
\tau_\theta = -P_m(a/r)^2 + (\tau_\phi/2)[1+(a/r)^2]+[1+3(a/r)^4] \cos 2 \theta 
\]

(15b)

\[
\tau_\phi = -(\tau_\phi/2)[1-3(a/r)^2]+2(a/r)^4] \sin 2 \theta 
\]

(15c)

The corresponding stress trajectory field is mapped in Figure 8c. This stress trajectory map arises when the wellbore pressure is much greater than the tectonic stress. Note that in the absence of a residual hydraulic pressure on the wellbore, (i.e. \( P_m = 0 \)), Eqs. (15a-c) simplify to exactly the Kirsch equations found by differentiating the stress function of Eq. (13). Note further that if \( (a/r)^2 \) approaches zero, Eqs. (15a-c) simplify to:

\[
\tau_r = (\tau_\phi/2)[1-\cos 2 \theta] = \tau_\phi \sin^2 \theta 
\]

(16a)

\[
\tau_\theta = (\tau_\phi/2)[1+\cos 2 \theta] = \tau_\phi \cos^2 \theta 
\]

(16b)

\[
\tau_\phi = (\tau_\phi/2) \sin 2 \theta 
\]

(16c)

which is the undisturbed regional background stress similar to that resulting from differentiating the stress function in Eq. (8b). However, \( (a/r)^2 \) approaches zero only if the radius ‘a’ of the borehole is very small and at large radial distance from the hole. Consequently we continue the analysis with the more complete Eqs. (15a-c).

Vertical drill holes with drilling mud, \( P_m \), supplied to equal the lithostatic pressure at each depth, will have no residual or effective deviatoric stress on the wall of the borehole from that mud column. Consequently, the use of the Kirsch [1898] equations is valid for such cases. However, for overbalanced and underbalanced borehole sections Eqs. (15a – c) are better descriptors. The stress pattern of Figure 8c arises for overbalanced mud columns, when formation pressure is low. For holes with under-pressure (i.e. \( P_m < 0 \)), due to over-pressured formations, or underbalanced mud columns, Figure 8e provides the proper stress pattern. It can also be seen that breakout is particularly likely to occur when \( P_m < 0 \), as the wellbore has the compressive principal stress parallel to its walls, and its magnitude is enhanced due to the under-pressurized borehole. The stress trajectory pattern of Fig. 8e has been also been mapped in photo-elastic stress analysis of a compressed platen with a cylindrical hole subject to a pressurized plug (Ramberg [47], figs. 4 & 5). Flattening of perforated photo-elastic discs confirmed the stress field of Fig. 8e [47].

9. NON-DIMENSIONAL EQUATIONS FOR HYDROFRACTURES

Eqs. (15a-c) may be non-dimensionalized for general application by defining the non-dimensionalized stress \( \tau^* \) (or \( \tau \) normalized by \( P_m \)), according to:

\[
\tau_r^* = \tau_r/P_m = \frac{\tau_\phi}{(4/r^2)+[1-(a/r)^2]-[1-4(a/r)^2]a^2 + 3(a/r)^4] \cos 2 \theta} 
\]

(17a)

\[
\tau_\theta^* = \tau_\theta/P_m = \frac{(\tau_\phi)}{[1+(a/r)^2]+[1+3(a/r)^4] \cos 2 \theta} 
\]

(17b)

\[
\tau_\phi^* = \tau_\phi/P_m = \frac{-(\tau_\phi)}{[1-3(a/r)^2]+2(a/r)^4] \sin 2 \theta} 
\]

(17c)

Introduction of the scaling parameter \( \xi = \tau_r/P_m \) in cylindrical space (\( r, \theta \)) with radial coordinates normalized according to \( r = a/r \) transforms Eqs. (17a-c) to:

\[
\tau_r^* = (1/r^2)\xi^*+[(1/r^2)][1-4(1/r^2)+3(3/r^2)] \cos 2 \theta 
\]

(18a)

\[
\tau_\theta^* = 0 
\]

(18b)

\[
\tau_\phi^* = 0 
\]

(18c)
The scaling parameter $\zeta$ is the non-dimensional ratio of the tectonic background stress $\tau_1$ and the hydraulic pressure, $P_m$, on the walls of the borehole.

\[
\tau_0^* = \frac{(1/r^2) + (\zeta/2)[1+(1/r^2)] + [1+(3/r^4)] \cos 2\theta}{1+(3/r^4)+2(r^2/2}) \sin 2\theta \quad (18b)
\]

\[
\tau_0 = -\frac{\zeta/2}{1-(3/r^4)+(2/r^2)} \sin 2\theta \quad (18c)
\]

The stress trajectories may now be mapped in $(r^*, \theta)$-space using various ratios of hydraulic pressure $P_m$ and regional far field stress $\tau_1$ as expressed in the scaling factor $\zeta$. The inclination $\beta$ of a stress trajectory with respect to the $r^*$-axes may be determined from:

\[
\tan 2\beta = 2\tau_0^*/(\tau_{r^*}-\tau_{\theta^*}) \quad (19)
\]

with two solution for $\beta$ separated by $\pi/2$. A continuous solution of Eq. (19) in a particular space outlines the stress trajectories.

The stress trajectory function of Eq. (19) has been evaluated numerically and some examples are mapped in Figures 9a-f. A scaling factor $\zeta=0.01$ implies that the compressive background stress $\tau_1=0.01 P_m$ so that the stress trajectories are mainly due to the hydraulic pressure on the borehole walls resulting in a nearly radial symmetric spider-web pattern (Fig. 9a). The other extreme case occurs when $\zeta$ is larger than unity (i.e., $\tau_1 >> P_m$), implying that the hydraulic pressure $P_m$ becomes negligible in comparison to the tectonic stress $\tau_1$ so that the stress trajectory pattern is nearly similar to that seen around an empty hole (compare Figs. 7 and 9f).

10. CONCLUSIONS AND RECOMMENDATIONS

Frac jobs are widely applied to enhance the production rate from wells drilled in so-called unconventional hydrocarbon source rocks with poor connectivity. Some formations may be sensitive to active deviatoric stresses due to tectonic compression or extension, commonly related to plate tectonics [48]. This study examined the interaction of the tectonic deviatoric stress and the hydraulic pressure during a frac job. The new stress function developed here provides analytical solutions for the state of stress around boreholes for all distinct cases, including the one described by the Kirsch equations (Fig. 10a-c). Analytical descriptions are transparent and preferred because they represent the simplest class of theoretical solutions.

Applying scaling theory showed that hydraulic cracks in the absence of tectonic background stress propagate only half a meter in response to commonly applied hydraulic pressures of up to 55 MPa. Observed fracture motion of up to 0.5 km from the hydraulically pressurized wellbore can be explained by dilation of pre-existing fractures with negligible yield strength. Pre-existing cracks can dilate and open over lengths up to 16 meter by a single pressure cycle. Further fracture propagation can be achieved by repeated hydraulic pressuring to allow for propagation of stress to the crack tip, which is confirmed by empirical evidence. The step-rate of the pressure cycle has little or no impact on the crack propagation process and may be set.
as high as practically achievable to complete the well stimulation job fastest from a cost point of view.

Another insight resulting from this study is that hydraulic frac jobs will deflect the stress trajectories around the pressurized borehole such that the tectonic background stress has little effect on the failure direction in the immediate vicinity of the borehole. A final practical recommendation is that steep pressure drops during the frac cycle must be avoided as such drops induce shear failure breakout of the borehole walls, which may obstruct the connectivity required for effective production. Shear failure can be prevented by lowering suction pressures and a slower flowback rate. In practice well completions involve perforations in the casing that will focus hydraulic pressure on predetermined sections of the borehole in the pay zone. High viscosity frac fluid such as gels will not only better carry proppants, but also buffer steep pressure drops and are therefore preferred when shear failure handicaps completion jobs.

The stress function developed here can also account for natural overpressures which may jeopardize drilling activities when the well bore crosses rock formations with extremely high or low formation pressures not balanced by the weight of the drilling mud column. The wall rock then frequently washes out and break outs damage the wellbore integrity. The 2010 Macondo well damage drilled by Deep Horizon for BP in the Gulf of Mexico is an example of an unbalanced wellbore. The simple analytical solutions developed here may help to prevent blow-outs and loss of drilling fluid in damaged formations; under-pressures are also accounted for by the analytical description. Better fracture placement is therefore possible in frac jobs, which typically account for 30% of well development cost, using the analytical solutions provided here using a comprehensive stress function. Such better fracture fairways then enhance well productivity and thus help bring down the cost per unit of recovered gas. This is now much needed to stem growing concerns of oil business analysts about the economic gap caused by depressed natural gas prices [49] – gas is currently sold below production cost by many unconventional gas companies in the US.

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**Fig. 10:** (a) Geological cross-section of drilling operation in sedimentary sequence with highly variable formation pressures. (b) The Kirsch [1] solutions of the stress field for the subsurface conditions in layer 3. (c) The stress function solutions of this study can model all potential situations. [$\zeta$ is a scaling factor; $P_m$ is the fluid pressure of the mud column in the borehole, and $\tau_1$, $\tau_2$, and $\tau_3$ are the principal deviatoric stresses in cylindrical coordinates $(R,\theta)$]. The potential tension joints are highlighted by thickened black lines and curves.
APPENDIX A: EQUILIBRIUM EQUATIONS

The differential equations of equilibrium governing the conditions outlined are obtained as follows. Figure A-1 illustrates the cross-section through the principal plane of a small volume of rock in equilibrium. Symbols used are cube faces 1, 2, 3 and 4, with rib lengths a, and b, as indicated. On account of stress gradients in the rock volume, the value of \( \tau_{zz} \) on faces 1 and 3 may vary slightly. Neglecting any body forces or including those in \( \tau_{xx} \), the equilibrium of forces in the X-direction is:

\[
\begin{align*}
bt_{(xx)1} + at_{(xx)2} + rt_{(xx)3} = 0
\end{align*}
\]  

(A1-a)

if divided by ab:

\[
\begin{align*}
\frac{t_{(xx)1}}{a} + \frac{t_{(xx)2}}{b} - \frac{t_{(xx)3}}{c} = 0
\end{align*}
\]  

(A1-b)

These are the differential equations of equilibrium for two-dimensional sections. The corresponding 3-D expressions in index notation can be summarized as:

\[
\begin{align*}
\partial_{xx}u + \tau_{xz} = 0
\end{align*}
\]  

(A2-a)

The equation of equilibrium for forces in the Z-direction is obtained similarly:

\[
\begin{align*}
\partial_{zz}w + \tau_{xz} = 0
\end{align*}
\]  

(A2-b)

Equation (A3) must be satisfied at all points in the rock volume. In geological applications superposed body force generated inside each particle due to their gravitational attraction are negligibly small as compared to the surface stresses experienced by the singular particles, and therefore may be neglected in most cases. The principle of equilibrium of forces is featuring not only in the derivation of stress functions for deforming and deformed elastic rocks, but also in that of stream functions for flow in viscous rock bodies.

APPENDIX B: COMPATIBILITY EQUATIONS

The condition of stress compatibility is complementary to the equilibrium assumption and provides an additional relationship between the components of stress in a body. The simplest fashion of derivation is to first obtain the condition of strain compatibility. The principle of strain compatibility follows from the simple physical reality that strains in the deformed configuration of a continuum exclude gaps and overlaps of matter. Components of the strain tensor therefore cannot be taken arbitrary as functions of the coordinates in the displacement field and there exists a certain relationship, termed the compatibility condition.

Any two-dimensional deformation is fully described by three strain components: normal strains \( e_{xx}, e_{zz} \), and engineering shear strain \( \gamma_{xy} = 2e_{yz} \). The strain may be expressed in terms of the gradients of displacement:

\[
\begin{align*}
e_{xx} &= \partial u / \partial x \\
e_{yz} &= \partial w / \partial z \\
\gamma_{xy} &= (\partial u / \partial z) + (\partial w / \partial x)
\end{align*}
\]  

(B1-a)

(B1-b)

(B1-c)

\[
\begin{align*}
\gamma_{xz} &= 2e_{yz} = 2e_{xz}
\end{align*}
\]  

These three equations can be combined after differentiating (B1-a) twice with respect to \( z \), (B1-b) twice with respect to \( x \), and (B1-c) once with respect to \( x \) and once with respect to \( z \):

\[
\begin{align*}
(\partial^2 e_{xx} / \partial z^2) + (\partial^2 e_{zz} / \partial x^2) &= 2\gamma_{xz} / \partial x \partial z \\
\end{align*}
\]  

(B2-a)

using \( \gamma_{xz} = 2e_{yx} \):

\[
\begin{align*}
(\partial^2 e_{xx} / \partial z^2) + (\partial^2 e_{zz} / \partial x^2) &= 2(\partial^2 e_{yz} / \partial x \partial z) \\
\end{align*}
\]  

(B2-b)

The differential equations (B2-b) is the condition of strain compatibility. Using a constitutive equation to link stress and strain, we find:

\[
\begin{align*}
(\partial^2 \tau_{xx} / \partial z^2) + (\partial^2 \tau_{zz} / \partial x^2) &= 2\gamma_{xz} / \partial x \partial z \\
\end{align*}
\]  

(B3-a)

or

\[
\begin{align*}
[(\partial^2 \tau_{xx} / \partial z^2) + (\partial^2 \tau_{zz} / \partial x^2)](\tau_{xx} + \tau_{zz}) &= 2\tau_{xz} / \partial x \partial z
\end{align*}
\]  

(B3-b)
Equation (B3-a) implies that the normal and shear displacements caused by the stress have to be internally compatible in the material considered. For the three-dimensional case, the condition of strain compatibility is made up of six independent expressions contained in the tensor equation:

\[
(\partial^2 \tau_{ij}/\partial x_m \partial x_n) + (\partial^2 \tau_{mn}/\partial x_i \partial x_j) = (\partial^2 \tau_{im}/\partial x_j \partial x_n) + (\partial^2 \tau_{jn}/\partial x_i \partial x_m) = 0 \tag{B4}
\]

For example, consider a block of rock subjected to a pure shear deformation, with boundary forces such that deviatoric stresses \( \tau_{xx} = -\tau_{zz} \) and \( \tau_{xz} = 0 \). This is a valid boundary condition, and there is no spatial gradient in any of the stress components. The equilibrium and compatibility conditions are fulfilled in any point of the block.

APPENDIX C: HENCKY’S THEOREMS FOR STRESS TRAJECTORIES AND SLIP CURVES

Hencky (see Hill [50]) has formulated two practical theorems related to some special properties of stress trajectories and slip-curve fields, after considering a curvilinear quadrilateral ABCD (Fig. C-1).

\[
\theta_B - \theta_A = \theta_C - \theta_D \tag{C1}
\]
\[
\theta_C - \theta_B = \theta_D - \theta_A \tag{C2}
\]

If orthogonal curvilinear coordinates are chosen in the node points of one of the corners of the quadrilateral, Hencky's first theorem achieves great practical value. The unknown angle \( \gamma \) follows from the known angles \( \alpha \) and \( \beta \):

\[
\beta = \theta_B + \theta_D \tag{C3}
\]

Hencky's second theorem refers to the curvature of slip curves as well as to that of sets of principal stress trajectories (Figure C-2). The curvature of a \( \beta \)-curve is given by the change in curvature of an \( \alpha \)-curve \( dR/\theta \) divided by the change in the angle of rotation \( \theta \) (Fig. C-2):

\[
R_{\beta} = dR/\theta \tag{C4}
\]

The above relationships are useful to check the validity of analytical and numerically derived solutions for stress trajectories and slip curves.

REFERENCES

44. Gough, D.I., and Bell, J.S., 1982. Stress orientations from borehole wall fractures with examples from


