Mapping stress trajectories and width of the stress-perturbation zone near a cylindrical wellbore

Ruud Weijermars a, b, *

a Department of Geoscience & Engineering, Delft University of Technology, Stevinweg 1, Delft 2628CN, The Netherlands
b Alboran Energy Strategy Consultants, Delft, The Netherlands

A R T I C L E   I N F O

Article history:
Received 21 August 2012
Received in revised form 13 March 2013
Accepted 12 August 2013
Available online 21 September 2013

Keywords:
Wellbore stress
Stress trajectories
Mud pressure
Stress-perturbation zone
Balanced boreholes

A B S T R A C T

This study reviews the analytical descriptions for stress characterization around balanced and unbalanced drill holes. The stress-trajectory patterns around cylindrical wellbores are visualized for a range of typical physical conditions. The effects of variations in far-field stress, boundary conditions, wellbore fluid pressure, and formation pressure are systematically outlined. Axially symmetric and asymmetric far-field stresses and their interaction with various wellbore pressures are quantified in diagrams scaled for universal use. The stress-perturbation zone, the region around the wellbore that is affected by a stress perturbation due to the presence of the wellbore, is delineated. Rules are formulated for practical application in wellbore-balancing studies and wellbore-stability analysis. These rules are useful for application in drilling activities aimed at the safe and effective extraction of energy resources (geothermal heat, oil, wet gas, dry gas).

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1. Introduction

Stresses around wellbores in rocks cannot be seen directly, but can be monitored during drilling operations and in experimental field-studies, as concisely reviewed in Ref. [1]. Present-day wellbore-stress monitoring-systems rely heavily on automated alarms for managing drilling pressure and maintaining wellbore stability [2–4]. The implementation of real-time data-flow-monitoring in boreholes improves process safety and enables the identification of challenges before they become bigger problems.

Wellbore-stability-control optimization benefits from a better insight resulting from wellbore-stress analysis. The classical expressions of Kirsch [5] have been used in many engineering applications of stressed plates with holes (e.g., [6–11]). The Kirsch equations also have been widely used as a basis to analytically model the state of stress around boreholes, assuming a homogeneous, isotropic and linear elastic rock continuum. Summaries and historic context of the adapted use of Kirsch equations in borehole problems have been outlined in several landmark studies (e.g., [1,12–14]).

In currently available borehole-stress monitoring-tools, little emphasis has been laid on the visualization of the stress trajectories around the wellbore. This is a gap in contemporary wellbore stress analysis: although principal stress trajectories and isobar patterns transparently and comprehensively characterize some of the most critical aspect of wellbore stresses, classical studies show only a standard image of qualitatively sketched stress trajectories for one special case—namely stress trajectories around an empty borehole subjected to an uniaxial far-field boundary stress (e.g., [10]). Zoback [13] reproduced such a trajectory pattern from [10], and quotes Kirsch [5] as the source of his Fig. 6.1, but the latter did not include any stress trajectory plot—as Fjaer et al. [14] noted, the original German language paper of Kirsch [5] is an example of a classic paper that is probably more often cited than read.

Recent studies [15–18] have shown a wide range of near-wellbore stress-trajectory patterns may occur depending on the boundary conditions for the far-field stress and the mud pressure applied to wellbores, even when the wellbore is kept aligned with one of the principal stress axes to maintain maximum symmetry in the analytical solutions. Stress algorithms are applied in software for drilling control, automated monitoring-systems, acoustic logging-devices, and hydraulic fracturing-tools. The merits of stress-analysis techniques are: (1) risk mitigation by anticipating adverse effects during drilling operations (blowout, stuck drill-string, wellbore damage, lost mud-circulation, mud-barrier failure and pressure kicks that activate blowout preventers), and (2) optimization of well performance after drilling (accurate hydraulic fracture placement, proper identification of far-field stresses on the basis of break-out observations, and the basis of inversion of acoustic signals).

The present study systematically reviews the existing array of analytical expressions for the most relevant borehole-stress conditions,
visualizes the pertinent stress-trajectories, and determines the width of the stress-perturbation zone due to the wellbore’s presence. Section 2 first considers the simple cases of pressured wellbores in radial symmetric far-field stress-regimes, illustrated with new graphs for practical use. Section 3 revisits the stressed hole subject to a uniaxial far-field stress originally considered by Kirsch [5], and its expansion to the distinct cases of wellbores penetrating rocks with a bi-axial far-field stress-state. Our synthetic wellbore simulates the effect of progressively higher wellbore pressures and shows that the classical stress-trajectory pattern for the uniaxial far-field tension described by the Kirsch equations is but one end-member in a continuous range of stress states in the plane of view normal to the wellbore axis. The isotropic or neutral points may move away from the wellbore when increasingly higher pressures are applied in so-called overbalanced boreholes. Section 4 provides practical rules for wellbore-stress analyses based on our analysis and highlights how to avoid common pitfalls. Section 5 outlines our conclusions.

2. Stress fields with axially symmetric solutions

2.1. Notation used in this study

Mechanical engineers traditionally use a positive sign for tensile stress in classical material science, and geotechnical engineers often take compressive stress as positive. The sign convention adopted here takes compressional stresses as positive. Irrespective of sign convention used, the physical meaning of wellbore pressures and stresses is adequately and consistently described by our symbols, subscripts and sign convention.

Assume a wellbore is planned aligned with the intermediate principal stress axis \( \sigma_2 \). The boundary conditions are, using cylindrical coordinates \((R, \theta)\) in a section normal to the future wellbore of radius \( R \):

\[
\sigma_R = \sigma_2 \quad \sigma_\theta = \sigma_3 \quad \sigma_0 = -\sigma_1
\]

We assume an incompressible elastic rock (constant density assumption for validity of the continuum assumption), but the continuum may possess some permeability to account for the presence of pore pressure within the formation. The unperturbed radial and tangential stresses on surfaces along the trajectory of a planned wellbore follow from the equations for the Mohr circle of stress (cf. [19]):

\[
\sigma_R = \frac{[(\sigma_1 + \sigma_3)/2] + [(\sigma_1 - \sigma_3)/2] \cos 2\theta}{(1 + \sin 2\theta)} \quad (1a)
\]
\[
\sigma_\theta = \frac{[(\sigma_1 + \sigma_3)/2] - [(\sigma_1 - \sigma_3)/2] \cos 2\theta}{(1 + \sin 2\theta)} \quad (1b)
\]
\[
\sigma_0 = \frac{[(\sigma_1 - \sigma_3)/2] \sin 2\theta}{(1 + \sin 2\theta)} \quad (1c)
\]

Table 1

<table>
<thead>
<tr>
<th>Tectonic Setting of Basin</th>
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</tr>
<tr>
<td>Trans-compression</td>
<td>( \sigma_V, \sigma_H, \sigma_H )</td>
</tr>
</tbody>
</table>

* For deviatoric stress conversions, use \( t \) instead of \( \sigma \) for all symbols used in this table.

We have chosen the notation to depart from the traditional approach in borehole stress analysis, which denotes a maximum total stress in the horizontal plane by \( \sigma_H \) (or \( \sigma_{H_{\text{MAX}}} \) or \( \sigma_H \)) and a minimum by \( \sigma_V \) (or \( \sigma_{H_{\text{MIN}}} \) or \( \sigma_V \)), and a vertical stress by \( \sigma_0 \) (or \( \sigma_H \)). The reason is that in many natural situations \( \sigma_H \) and \( \sigma_V \) are not principal stresses at all, while Eqs. (1a)–(1c) are only valid for principal stresses. Our notation, using \( \sigma_1, \sigma_2, \) and \( \sigma_3 \), is arguably more practical when matching solutions to the natural range of tectonic regimes [16]. Table 1 gives a conversion table for relating our subscript choice to Andersonian stress regimes, amplified with transpressional and trans-tensional cases.

In our paper, total stress is denoted by \( \sigma \) and deviatoric stress by \( t \). Deviatoric stresses differ from the total stress in that isotropic stresses (pressure components), which do not contribute to deformation gradients, are taken out. The deviatoric stress can be represented in both a tensor and principal form. Our notation is in compliance with common practice in structural geology literature [20,21]. There are three principal deviatoric stresses \( \sigma_1, \sigma_2, \) and \( \sigma_3 \), of which—in our analysis—one of them is assumed to be aligned with the drilling direction; this fixes the two other principal stresses in the transverse direction. We use the physical effects of principal stresses, i.e., compression and tension, to fix the subscripts. Compressional stresses are in our study consistently denoted by \( \sigma_1 \) and tensional stresses by \( \sigma_3 \) in compliance with the prevailing annotation in structural geology and tectonic models. In axially symmetric far-field stresses, with bi-axial stresses \( \sigma_1 \) and \( \sigma_3 \), \( \sigma_1 \) may be equal to \( \sigma_3 \), in which case they are both either tensile or compressional stresses.

2.2. Wellbore stresses around an empty hole with far-field axially symmetric tension (no pressure on wellbore)

Kirsch [5] was in his days mostly interested in investigating how a far-field stress gives rise to stress concentrations around cylindrical holes in elastic thick plates (which justifies a 2D analysis). He first assumed an axially symmetric far-field tensional stress, \( \tau_0 \) (Fig. 1a). The boundary conditions are: \( \tau_R = 0 \) at \( R = \infty \); \( \tau_\theta = 0 \) at \( R = a \); \( \tau_z = 0 \) at \( R = \infty \); and \( \tau_z = \tau_\theta = 2\tau_0 \) at \( R = a \). These conditions impose a radial tension, \( \tau_R \), and a tangential tension, \( \tau_\theta \) (Fig. 1b), with the magnitudes at radial distance, \( R \), affected by the presence of the hole with radius, \( a \), as follows:

\[
\tau_R = \tau_0 [1 - (a/R)^2] \quad (2a)
\]
\[
\tau_\theta = \tau_0 [1 + (a/R)^2] \quad (2b)
\]

For strain compatibility and stress balance, \( \tau_R + \tau_\theta = -\tau_z \).

Eqs. (2a) and (2b) are plotted in Fig. 2a and b, which shows that the radial stresses, \( \tau_R \), completely vanish at the boundary of the empty hole (henceforth \( \tau_R = 0 \) at \( R = a \)). At a distance of three times the borehole radius \( a \) (\( R = 3a \)), \( \tau_R \) still retains 90% of its far-field value \( \tau_0 \). In contrast, the tangential stress, \( \tau_\theta \), increases toward the hole and reaches a maximum value of twice the far-field stress \( \tau_\theta = 2\tau_0 \), which mean the so-called stress-concentration factor is 2 at \( R = a \). Both the tangential and radial stresses resume nearly equal magnitudes (resp. 103% and 97%) at six times the hole radius distance \( R = 6a \), see Fig. 2a and b), which approaches the outer limit of the near-wellbore zone with an unperturbed far-field stress state.

The principal stress trajectories are given by the cobweb pattern of Fig. 1b, and stress magnitudes are scaled in Fig. 2b.

2.3. Wellbore stresses due to mud load only (no far-field stress on wellbore)

The assumption of an empty borehole with zero-pressure on the wellbore is not applicable to most practical drilling situations.
Instead we consider a mud-filled borehole (Fig. 3a) where the traction exerted on the wellbore is a radial compression \( P_M \), equal to the wellbore mud-pressure \( P_M = \rho M g h \) (Fig. 3b). Pressure itself is a scalar quantity, but induces a force vector on the wellbore. The boundary conditions are:

- \( \tau_R = 0 \) at \( R = \infty \); \( \tau_R = P_M \) at \( R = a \);
- \( \tau_\theta = 0 \) at \( R = \infty \); and \( \tau_\theta = -P_M \) at \( R = a \). For this case, the stress equilibrium Eqs. (2a) and (2b) modify to (see [15]):

\[
\begin{align*}
\tau_R &= \frac{P_M (a/R)^2}{C_0} \\
\tau_\theta &= \frac{P_M (a/R)^2}{C_0}
\end{align*}
\]

The decay of stress away from such overbalanced (over-pressured) boreholes is plotted in Fig. 4a and b. The principal stress trajectories are given by the cobweb pattern of Fig. 3b, and stress magnitudes are scaled in Fig. 4b.

2.4. Wellbore stress around boreholes with mud pressure and far-field tension

The stresses around a wellbore with mud load (case of Sections 2.3) as well as a far-field axially symmetric tension stress (case of Sections 2.2) can be combined to give rise to the following boundary conditions:

- \( \tau_R = \tau_0 \) at \( R = \infty \); \( \tau_R = P_M \) at \( R = a \);
- \( \tau_\theta = \tau_0 \) at \( R = \infty \); and \( \tau_0 = 2\tau_0 - P_M \) at \( R = a \). The stress state around the wellbore can be modeled by summing Eqs. (2a) and (2b) and Eqs. (3a) and (3b):

\[
\begin{align*}
\tau_R &= \tau_0 + (P_M - \tau_0)(a/R)^2 \\
\tau_\theta &= \frac{\tau_0 + (a/R)^2 - P_M (a/R)^2}{C_0}
\end{align*}
\]

Evaluation of Eqs. (4a) and (4b) shows that when the far-field stress \( \tau_0 \) and the normal compression from the mud-balancing pressure \( P_M \) are equal in strength (but with opposite sign, \( P_M = -\tau_0 \)), the
radial stress \( \tau_R \) becomes a compressive stress at the wellbore equal to 100% the absolute magnitude of the far-field stress \(|\tau_0|\) (as compared to 0% in the absence of a wellbore pressure for the case of Sections 2.2, Fig. 2a). Due to the mud pressure, the tangential stress becomes an enhanced tension equal to 300% the magnitude of the far-field tension \( \tau_0 \) (Fig. 5a, as compared to 200% in the absence of a wellbore pressure, see Fig. 2a). The stress-perturbation zone near the wellbore for the case \( P_M = \tau_0 \) is plotted in Fig. 5a and the corresponding stress trajectories are plotted in Fig. 5b.

2.5. What is a balanced wellbore?

It is sometimes (wrongly) assumed in drilling operations that any residual stress on the wellbore resulting from a far-field stress will vanish if \( P_M = \tau_0 \) (Fig. 5a), as compared to 0% in the absence of a wellbore pressure for the case of Sections 2.2, Fig. 2a). Due to the mud pressure, the tangential stress becomes an enhanced tension equal to 300% the magnitude of the far-field tension \( \tau_0 \) (Fig. 5a, as compared to 200% in the absence of a wellbore pressure, see Fig. 2a). The stress-perturbation zone near the wellbore for the case \( P_M = \tau_0 \) is plotted in Fig. 5a and the corresponding stress trajectories are plotted in Fig. 5b.

radial stress \( \tau_R \) becomes a compressive stress at the wellbore equal to 100% the absolute magnitude of the far-field stress \( 1 - \tau_0 \) (Fig. 5a), as compared to 0% in the absence of a wellbore pressure for the case of Sections 2.2, Fig. 2a). Due to the mud pressure, the tangential stress becomes an enhanced tension equal to 300% the magnitude of the far-field tension \( \tau_0 \) (Fig. 5a, as compared to 200% in the absence of a wellbore pressure, see Fig. 2a). The stress-perturbation zone near the wellbore for the case \( P_M = \tau_0 \) is plotted in Fig. 5a and the corresponding stress trajectories are plotted in Fig. 5b.

and at \( R=\infty \) we have the unperturbed far field stress \( \tau_0 = \tau_R = \tau_0 \). This shows that for \( P_M = -2\tau_0 \), the radial and tangential stresses at the wellbore will both increase (Fig. 6a) and the situation worsens for \( P_M = -3\tau_0 \) (Fig. 6b). The conclusion is that a far-field axially symmetric tension-stress cannot be balanced by applying any mud pressure to the wellbore. Axially symmetric tension stress occurs above expanding salt domes. The high tangential well stress cannot be balanced, unless a suction is applied such that \( P_M = 2\tau_0 \) or \( 3\tau_0 \). Although suction pumps are not employed for balancing purposes in regular drilling operations, the extreme environment of a far-field radial tension stress may need some experimenting in this direction. Applying a mud-pressure certainly has an adverse effect on wellbore stability and open-hole completion would be a better alternative.

2.6. Wellbore stress around boreholes with mud pressure and far-field compression

If we now consider a far field compression on the wellbore, such as due to lithostatic loading, we have at \( R=\infty \) a uniform compression \( \tau_R = \tau_0 = \rho g z / (1 - \nu) \) with Poisson ratio \( \nu \), gravity acceleration \( g \), overburden depth \( z \), and overburden density \( \rho \).
and tangential stresses rendered equal to \( f \), where the formation compression due to the lithostatic pressure (Fig. 7).

If we now take \( R = a \) at the wellbore, the radial and tangential stresses \( \tau_R = \tau_\theta \) according to Eqs. (6a) and (6b) are rendered equal to \( \rho g z_r (1-\nu) \), which is the unperturbed far-field compression due to the lithostatic pressure (Fig. 7).

A different situation arises when drilling into permeable rock where the formation fluid exerts a resultant pressure \( P_f \) due to the lithostatic load and can directly transfer this fluid pressure to the wellbore and impact the mud pressure \( \bar{P}_M \). This situation seems classical, but a careful discussion is merited. Two basic cases can be distinguished, with a static wellbore net-pressure (Fig. 8a) and a dynamic net-pressure (Fig. 8b). A static net-pressure requires a casing or a mud cake that effectively keeps the mud fluid and the formation fluid apart (Fig. 8a). In the case of a dynamic net-pressure, the two fluids communicate and the mud pressure in the wellbore will attain a value close to the formation pressure after some time (Fig. 8b). If there is permeable contact between the fluid in the drillhole (due to the mud weight, \( P_M \)), and the formation fluid (\( P_f \)), any pressure increase from the overburden thickness and/or tectonic forces will be communicated to the wellbore pressure.

For all cases, the wellbore net-pressure, \( P_{\text{NET}} \), is the unbalanced part of the wellbore tractions induced by the pressure differential, given by:

\[
P_{\text{NET}} = \bar{P}_M + \bar{P}_F = PM - PF
\]

\( P_M \) is the hydraulic traction on the wellbore due the mud-load pressure on the wellbore, and \( \bar{P}_F \) is the traction (with opposite sign) due to the formation pressure. The stress-balance equations result in the following expressions:

\[
\tau_R = [\rho g z_r (1-\nu)] + [(P_{\text{NET}} - \rho g z_r (1-\nu))(a/R)^2]
\]
Underbalanced open-hole completions have \( \rho \ll \rho_d \), which is why \( P_{\text{NET}} \) will then always be negative (Fig. 8a and b, bottom row).

Consequently, the situation of Fig. 7 cannot be achieved in underbalanced wellbores and instead the net pressure \( P_{\text{NET}} \) becomes a traction that contracts the wellbore surface inward, and induces a tensional normal stress at the wellbore wall (at \( R = a \)) with the resulting radial stress \( \tau_R = P_{\text{NET}} \) and \( \tau_\theta = 2 \rho g z / (1 - \nu) - P_{\text{NET}} \) (with \( P_{\text{NET}} < 0 \)). This corresponds to the situation in Fig. 5a, but with a sign reversal: radial stresses are tensional and tangential stresses are compressional. Remember that in all commercial gas reservoirs, the average reservoir pressure must be greater than the wellbore pressure—this is a prerequisite for a given amount of gas in place to be recovered. During the process of completion, there may be also brief intervals of time when the well remains underbalanced at the reservoir depth. This is why underbalanced well completions require a significant amount of specific equipment at the well site to ensure safe operations.

For balanced well completions, \( P_M \) is engineered with mud loads to aim for \( P_{\text{NET}} \) close to zero at most depths (Fig. 8a and b, middle cases), which would results again in the stress state plotted in Fig. 7. However, \( P_f \) is highly variable in natural rocks and may be difficult to balance at each specific level. Wellbores therefore are commonly engineered to be over-balanced during the construction process by selecting the mud weight such as to provide a positive wellbore net-pressure in the wellbore at all depths. The minimum mud-weight is therefore limited by the highest pressure (in gradient terms) and all of the rest of the exposed formations will then experience a significant overbalance. This is primarily a well-control (and safety) issue. The hydrostatic
pressure of the wellbore fluid is the “primary barrier” between the formations fluids (potentially hydrocarbons) and the surface.

For the overbalanced sections of the wellbore, \( P_{NET} \) will push the wellbore surface outward and thereby exerts a compression stress normal to the wellbore wall. This is similar to the situation portrayed in Fig. 6a and b. The response of the host rock is elastic distortion, until failure is initiated when the critical tensile stress normal to the wellbore wall. This is similar to the situation at bi-axial plane-stress. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

### 3. Asymmetric stress-field solutions

#### 3.1. Uni-axial far-field stress around an empty borehole

Kirsch [5] derived (using the contemporary textbooks by Bach [22] and Föppl [23]) a concise set of analytical expressions for the principal deviatoric radial stress \( (\tau_R) \) and principal tangential stress \( (\tau_\theta) \) around a hole in a thick plate subjected to a uni-axial far-field tension stress (Fig. 9a). The boundary conditions are: \( \tau_R = 0 \) at \( R = a \) and \( \infty \) for \( \theta = 0^\circ \); \( \tau_R = \tau_\theta \) at \( R = \infty \) for \( \theta = 90^\circ \); \( \tau_R = 0 \) at \( R = a \) for \( \theta = 90^\circ \); \( \tau_\theta = 0 \) at \( R = a \) for \( \theta = 0^\circ \); \( \tau_\theta = 0 \) at \( R = a \) and \( \infty \) for \( \theta = 90^\circ \); \( \tau_R = 0 \) at \( R = a \) for \( \theta = 90^\circ \). The Kirsch [5] equations for uni-axially stretched thick perforated plates have been widely used as a basis to analytically model the state of stress around boreholes:

\[
\tau_R = \frac{(\tau_\theta)}{2}\left[1-(a/R)^2\right] + \frac{1}{2}\left[1+3(a/R)^2\right] \cos 2\theta \quad \text{(9a)}
\]

\[
\tau_\theta = \frac{(\tau_\theta)}{2}\left[1+(a/R)^2\right] - \frac{1}{2}\left[1+3(a/R)^2\right] \sin 2\theta \quad \text{(9b)}
\]

\[
\tau_{\theta 0} = -\frac{(\tau_\theta)}{2}\left[1-(a/R)^2\right] + \frac{1}{2}\left[1+3(a/R)^2\right] \sin 2\theta \quad \text{(9c)}
\]

These are his classical expressions subsequently used in many engineering applications (e.g., [6–11]) and adapted for use in borehole problems (e.g., [1, 12–14]). It can also be readily seen that for \( \theta = 0^\circ \), Eqs. (9a) and (9b) become equal to Eqs. (2a) and (2b).

The far-field stress assumption by Kirsch [5], for the thick elastic plate containing a traction-free hole under uni-axial tension, \( \tau_\theta \), implies the boundary condition at \( r \to \infty \) is given by \( \tau_R = \tau_\theta \) for \( \theta = 0^\circ \) and \( \tau_R = 0 \) for \( \theta = 90^\circ \) (Fig. 9a). Essentially this means that the far-field direction normal to the uni-axial tension is a stress-free (rollers or free-hanging) boundary.

Kirsch [5] did not plot the principal stress trajectories. For the uni-axial tension, the stress trajectories are no longer axisymmetric and these therefore depart from the spiderweb patterns in Section 3 made up by radial and tangential stresses. The trajectories of the principal stresses may be mapped in \((R, \theta)\)-space, using Eqs. (9a)–(9c) to determine the inclination \( \beta \) of any spatial stress trajectory with respect to the \( R \)-axes from the following expression (see [15]):

\[
\tan 2\beta = 2\tau_{\theta 0}/(\tau_R - \tau_\theta) \quad \text{(10)}
\]

with two solution for \( \beta \) separated by \( \pi/2 \). A continuous solution of Eq. (10) in a particular space outlines the \( \tau_1 \) and \( \tau_3 \) stress trajectories. Fig. 9b shows the classical case of stress trajectories around an empty hole with a uni-axial far-field tension stress \( \tau_0 \). The stress-concentration factor is two times the far-field stress.

The magnitudes of the principal deviatoric stresses, \( \tau_R \) and \( \tau_\theta \), follow from Eqs. (9a)–(9c) based on knowledge of the far-field stress magnitude, \( \tau_0 \), in the location of each point determined by polar coordinates \((R, \theta)\), with radial distance, \( R \), drill hole with radius, \( a \), and angular position \( \theta \). Fig. 10a shows the stress isobar map for the uni-axial stress boundary condition. The isobars for the reactive stress are outlined in Fig. 10b. It is clear that tensile failure will occur in the direction perpendicular to the tension at the locations where the tensile stress reaches maxima of up to 2 times the far field tension, at the wellbore margin along the horizontal surface at the bottom of Fig. 10a. Kirsch [5] included...
experimental data of plates uni-axially stretched until failure, which persistently showed the plates developed a tension fracture at either side of the holes in the predicted stress concentration region.

3.2. Uni-axial far-field stress with overbalanced mud pressure

The magnitude of the elements of the stress tensor for a uni-axial stress field with a borehole net pressure in polar \((R, \theta)\)-space is (for details see [15]):

\[
\tau_R = P_{\text{NET}}(a/R)^2 + (\tau_0/2)[1-(a/R)^2][1-4(a/R)^2 + 3(a/R)^4] \cos 2\theta
\]

(11a)

\[
\tau_\theta = -P_{\text{NET}}(a/R)^2 + (\tau_0/2)[1+(a/R)^2][1+3(a/R)^4] \cos 2\theta
\]

(11b)

\[
\tau_{R\theta} = -(\tau_0/2)[1-3(a/R)^4 + 2(a/R)^2] \sin 2\theta
\]

(11c)

The boundary conditions are: \(\tau_R = P_{\text{NET}}\) at \(R=a\) for \(\theta=0^\circ\) and \(90^\circ\); \(\tau_R = 0\) at \(R=\infty\) for \(\theta=90^\circ\); \(\tau_R = \tau_0\) at \(R=\infty\) for \(\theta=0^\circ\); \(\tau_\theta = -P_{\text{NET}}\) at \(R=a\) for \(\theta=90^\circ\); \(\tau_\theta = 0\) at \(R=\infty\) for \(\theta=0^\circ\). Introduction the non-dimensional stresses \(\tau_R^* = \tau_R/P_{\text{NET}}, \tau_\theta^* = \tau_\theta/P_{\text{NET}}, \tau_{R\theta}^* = \tau_{R\theta}/P_{\text{NET}}\) and the non-dimensional scaling parameter \(F = P_{\text{NET}}/\tau_0\) in polar coordinate space \((R^*, \theta^*)\) with radial coordinates normalized according to \(R^* = R/r\) a transforms Eqs. (11a)-(11c) to their non-dimensional form [15]:

\[
\tau_R^* = (1/R^2) - (1/2F)[1-(1/R^2)]/[1-(4/R^2) + (3/R^4)] \cos 2\theta
\]

(12a)

\[
\tau_\theta^* = - (1/R^2) - (1/2F)[1 + (1/R^2)]/[1 + (3/R^4)] \cos 2\theta
\]

(12b)

The principal stress trajectory pattern in \((R^*, \theta^*)\)-space is fully determined by the inclination \(\beta\) of a stress trajectory with respect to the \(R\)-axes [15]:

\[
\tan 2\beta = 2\tau_{R\theta}^*/(\tau_\theta^* - \tau_R^*)
\]

(13)

with the two conjugate solutions for \(\beta\) separated by \(\pm\pi/2\). A continuous solution of Eq. (13) outlines the stress trajectories in a spatial plane perpendicular to the wellbore. The principal stress trajectories are mapped for the continuous range of all possible ratios of hydraulic pressure \(P_{\text{NET}}\) and regional far field stress \(\tau_0\) and substituted in the scaling factor, \(F\) in Eqs. (12a)-(12c) and (13).

The principal stress trajectory patterns for overbalanced wellbores—which occur when the \(F\) number becomes progressively positive—are mapped in Fig. 11a-d. The result is that for larger \(F\) numbers, all radial stresses are compressional—even in the direction of the far field tension stress. The two aligned principal stresses \(\tau_1\) (compression near the wellbore) and \(\tau_3\) (equal to the far field tension \(\tau_0\)) are separated by the isotropic or neutral points (red dots). Within the region outlined by the \(\tau_2\)-elliptical stress trajectory between the two neutral points, tension fractures can form only in the radial planes and curved surfaces that follow \(\tau_2\)-trajectories. Due to increasing positive \(P_{\text{NET}}\) in overbalanced wellbores, the directions of \(\tau_1\) and \(\tau_3\) (largest and smallest principal stresses) interchange near the wellbore (Fig. 11b). This effect may have been overlooked in wellbore breakout studies.

These results have important implications for hydraulic fracturing operations: as the hydraulic pressure increases \(P_{\text{NET}}\) increases and therefore the \(F\) number will rapidly shoot up. The resulting stress trajectories for heavily overbalanced wellbores imply that tension fractures will open in radial directions without a clear preferential direction.

3.3. General bi-axial far-field stress around an empty borehole

A generic solution of the force balance equation for the state of stress in wellbores subjected to a bi-axial tectonic background stress (Fig. 12a and b) can be obtained by the superposition of two stress functions \(\Phi_1\) for the far-field uni-axial stress in the \(X\)-direction, and \(\Phi_2\) for the far-field uni-axial stress in the \(Y\)-direction. These stress functions are simple summations of the Airy stress function \(\Phi(R, \theta)\) for a non-hydraulically pressurized, circular hole in a plate subjected to a uni-axial far-field stress, \(\tau_0\) [7]:

\[
\Phi(R, \theta) = \frac{(\tau_0/4)R^2 - (\tau_0/2)\alpha^2 \ln (R/a) + [(\tau_0/4)R^2 - (\tau_0/2)\alpha^2 + (\tau_0/4)\alpha^4/R^2]}{\alpha^4} \cos 2\theta
\]

(14)

The stress functions \(\Phi_1(R, \theta)\) and \(\Phi_2(R, \theta)\) for the two perpendicular uni-axial far-field stresses \(\tau_1\) and \(\tau_3\) (Fig. 11a) may be summed to yield a generic bi-axial stress function:

\[
\Phi_{\text{total}}(R, \theta) = \Phi_1(R, \theta) + \Phi_2(R, \theta) = (\tau_1/4)R^2 - (\tau_1/2)\alpha^2 \ln (R/a)
\]

\[
+ [(\tau_1/4)R^2 - (\tau_1/2)\alpha^2 + (\tau_1/4)\alpha^4/R^2] \cos 2\theta
\]

\[
+ (\tau_3/4)R^2 - (\tau_3/2)\alpha^2 \ln (R/a)
\]

\[
- [(\tau_3/4)R^2 - (\tau_3/2)\alpha^2 + (\tau_3/4)\alpha^4/R^2] \cos 2\theta
\]

(15)

The boundary condition at \(r \to \infty\) is given by \(\tau_R = \tau_1\) for \(\theta=0^\circ\) and \(\tau_R = \tau_0\) for \(\theta=90^\circ\). The radial and tangential normal stresses and symmetric shear stresses follow from differentiation of the stress function \(\Phi_{\text{total}}(R, \theta)\) according to Eqs. (16a)-(16c):

\[
\tau_R = -(\partial/\partial R)(1/R)(\partial \Phi/\partial \theta)
\]

(16a)

\[
\tau_\theta = \alpha^2 \partial^2 \Phi/\partial \theta^2
\]

(16b)

\[
\tau_{R\theta} = -\partial \Phi/\partial R
\]

(16c)
Application to Eq. (15) yields the generic expressions for stresses around a borehole subject to the conjugate far field stresses $\tau_1$ and $\tau_2$:

$$
\tau_R = \frac{[(\tau_1 + \tau_2)/2][1-(a/R)^2]+[(\tau_1 - \tau_2)/2][1-(a/R)^2][1-3(a/R)^2]}{\cos 2\theta}
$$

(17a)

$$
\tau_\theta = \frac{[(\tau_1 + \tau_2)/2][1+(a/R)^2]-[(\tau_1 - \tau_2)/2][1+3(a/R)^2]}{\cos 2\theta}
$$

(17b)

$$
\tau_\theta = -\frac{[(\tau_1 - \tau_2)/2][1-(a/R)^2][1+3(a/R)^2]}{\sin 2\theta}
$$

(17c)

The magnitudes of the principal deviatoric stresses, $\tau_\theta$ and $\tau_\theta$, follow from Eqs. (17a)–(17c) based on knowledge of the far-field stress magnitude, $\tau_1$ and $\tau_2$, in the location of each point determined by polar coordinates $(R, \theta)$, with radial distance, $R$, and drill hole with radius, $a$, and angular position $\theta$. One can see that at the borehole radius the radial stress vanishes, i.e., $\tau_R = 0$ at $(a/R) = 1$. The tangential stress-concentration factor at the wellbore at $R=a$ for $\theta=0$ is $\tau_\theta = 3\tau_1 - \tau_2$.

Adopting a plane stress assumption, $\tau_\theta = -\tau_1$ (at $R \rightarrow \infty$), simplifies Eqs. (17a)–(17c) to (valid only for $R \geq a$):

$$
\tau_R = \tau_1[1-4(a/R)^2+3(a/R)^4] \cos 2\theta
$$

(18a)

$$
\tau_\theta = -\tau_1[1+3(a/R)^4] \cos 2\theta
$$

(18b)

$$
\tau_\theta = -\tau_1[1+2(a/R)^2-3(a/R)^4] \sin 2\theta
$$

(18c)

This solution differs from Eqs. (9a)–(9c), which is for a uni-axial far-field stress and Eqs. (18a)–(18c) are for a bi-axial (plane) far-field stress. The principal stress trajectories around the pressure-free hole due to a far-field plane stress have been visualized in Fig. 12b. The tangential stress-concentration factor at the wellbore is 4 times $\tau_1$ at $R=a$ for $\theta=0$, i.e., $\tau_\theta = 4\tau_1$. Radial tension fractures form when the tangential (hoop) stress is tensional and exceeds the rock’s tensile strength, $\tau_\theta$, according to $\tau_\theta \geq \tau_c$.

### 3.4. Bi-axial far-field stress with mud pressure

The balancing effect of any fluid pressure on the wellbore can be accounted for by including in Eqs. (17a)–(17c)—analogous to expressions (11a–11c)—the net pressure:

$$
\tau_R = \frac{P_{net}(a/R)^2+[\tau_1 + \tau_2]/2][1-(a/R)^2]+[(\tau_1 - \tau_2)/2][1-(a/R)^2][1-3(a/R)^2]}{\cos 2\theta}
$$

(19a)
Stephansson [1], which is based on the work of Scheidegger [24]: The implied boundary conditions for the stresses at the borehole mud-barrier failure due to pressure kicks and may lead to too low for the pressure in certain formations increase the risk of underbalanced, rather than overbalanced, mud pressures that are net-pressure in the wellbore does not exceed the pore pressure or net-pressure in the borehole. The reference to color in this blue curvi-linears. Compressional stresses are aligned with the red trajectories. Isotropic points emanating from the wellbore wall are 90° apart. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

\[
\begin{align*}
\tau_0 &= -P_{NET}(a/R)^2 + [(r_1 + r_3)/2][1 + (a/R)^2 - (r_1 - r_3)/2][1 + 3(a/R)^4] \cos \theta \\
\tau_{R0} &= -[(r_1 - r_3)/2][1 - (a/R)^2][1 + 3(a/R)^4] \sin \theta
\end{align*}
\]  \hspace{1cm} (19b)

The implied boundary conditions for the stresses at the borehole are: \(\tau_R = P_{NET}\) at \(R = a\); \(\tau_0 = r_1\) at \(R = \infty\) for \(\theta = 0\); \(\tau_0 = r_3\) at \(R = \infty\) for \(\theta = 90\°\); \(\tau_0 = 3r_3 - r_3 - P_{NET}\) at \(R = a\) for \(\theta = 0\); \(\tau_0 = -P_{NET}\) at \(R = a\) for \(\theta = 90\°\); \(\tau_0 = r_1\) at \(R = \infty\) for \(\theta = 90\°\); \(\tau_0 = r_3\) at \(R = \infty\) for \(\theta = 0\).

Someone with ordinary skill in geo-mechanics can prove that the case for net pressured wellbores described by Eqs. (19a)-(19c) become identical to the generally accepted summary format of Zang and Stephansson [1], which is based on the work of Scheidegger [24]:

\[
\begin{align*}
\sigma_{RR} &= (1 - \rho^2)(S_H + S_b)/2 + (1 + 4\rho^2 + 3\rho^4)(S_H - S_b)/2 \cos \Delta \rho \rho^2 \\
\sigma_{R\theta} &= (1 + \rho^2)(S_H + S_b)/2 - (1 + 3\rho^2)(S_H - S_b)/2 \cos \Delta \rho \rho^2 \\
\sigma_{\theta\theta} &= -[(1 + 2\rho^2 - 3\rho^4)(S_H - S_b)/2] \sin \Delta \rho \rho^2
\end{align*}
\]  \hspace{1cm} (20a)

(20b)

(20c)

with \(\rho = a/R\) and \(\Delta \rho = \rho - \rho\) the net-pressure in the borehole. The reference frame uses polar coordinates \((R, \theta)\), and two stresses \(S_H\) and \(S_b\) acting at infinity correspond to \(r_1\) and \(r_3\) used in our approach. One can reconcile the expression of Zang and Stephansson [1] with the original Kirsch equations (Eqs. (9a)–(9c) in our study) by taking \(S_b = 0\) and \(\Delta \rho = 0\) (Kirsch [5] did not consider over- or under-pressured holes).

4. Discussion and practical rules

4.1. State-of-the-art gaps

Wellbore integrity is compromised when the safe mud-weight window is overstepped. Wellbore pressures are “safe” when the net-pressure in the wellbore does not exceed the pore pressure or (formation pressure) as this would lead to permanent loss of mud-circulation fluid—which is costly. If well sections are drilled underbalanced, rather than overbalanced, mud pressures that are too low for the pressure in certain formations increase the risk of mud-barrier failure due to pressure kicks and may lead to blowouts. High residual-stress and high stress concentration-factors, due to incomplete balancing of the wellbore stresses by the mud-weight, in either under- or over-balanced drilling, may lead to tensile failure and shear failure (breakout).

With drilling operations moving to HP/HT conditions in ever deeper wells [25–27], the theoretical models of drilling risk and uncertainty envelopes must become more sophisticated. Reservoirs with riskier pore pressure and fracture-gradient windows require sophisticated stress analysis to ensure the well remains within the margins of the safe-drilling window. Pre-drilling decisions about drill-hole-path-trajectory planning and the associated wellbore-stability analysis will benefit from the inclusions of stress-trajectory visualization around the full length of the wellbore. Our present analysis aims to contribute to the development of improved stress-analysis tools for inclusion in contemporary monitoring tools an automated dash-boards or wellbore-stability analysis.

Basic analytical equations for wellbore-stress quantification have been available and adapted for use in borehole stability analysis since the ground-breaking work of Kirsch [5]. However, this has not translated into a comprehensive visualization of wellbore-stress trajectories for a range of far-field and inner-wellbore boundary-conditions. Our study intends to fill part of this gap and a systematic analysis revealed a number of worthwhile practical implications articulated below.

4.2. Practical rules

Numerous practitioners use the following set of “generic” expressions to estimate the radial and tangential stresses at pressured vertical wellbores with an assumed far-field stress \(\sigma_H\) and \(\sigma_b\) (e.g., [28]; correcting for a sign reversal in \(\sigma_R\)):

\[
\begin{align*}
\sigma_R &= P \\
\sigma_\theta &= 3\sigma_H - \sigma_b - P \\
\sigma_{R\theta} &= 0
\end{align*}
\]  \hspace{1cm} (21a)

(21b)

(21c)
The adoption of such a generic set of equations introduces several sources of potential inaccuracy in wellbore-stress analysis:

(A1) Total stresses \( \sigma_{B} \) and \( \sigma_{P} \) may at depth no longer be aligned with principal stresses. Use of \( \sigma_{1}, \sigma_{2}, \) and \( \sigma_{3}, \) (and corresponding principal deviatoric stresses \( \tau_{12}, \tau_{23}, \) and \( \tau_{31} \)) is arguably more practical. In any case, when \( \sigma_{B} \) and \( \sigma_{P} \) are used assurances are needed on how these relate to the principal stress directions. If stresses are no longer Andersonian (see Section 2.1) at depth (as in most natural deformations, especially when drilling near salt bodies [29–34]), the tensor transformation rules should be used to relate \( \sigma_{B} \) and \( \sigma_{P} \) to the principal stresses acting on the wellbore.

(A2) The stress-concentration factor of 3 times the maximum far-field stress implied by Eq. (21b) is only valid for a bi-axial far-field stress [described in our study by Eqs. (17a)–(17c)]. The stress-concentration factor for the special case of a bi-axial far-field plane-stress is 4 times the maximum far-field stress [described in our study by Eqs. (18a)–(18c)]. If a far-field stress is axially symmetric [such as outlined in Eqs. (2a)–(2c)] the stress-concentration factor is 2 times the far-field stress. This distinction is of practical importance as such far-field stresses occur above expanding salt domes. The same stress concentration factor of 2 applies to wellbores penetrating salt-weld controlled depo-centers, which are situated in radial symmetric far-field compression stress (assuming these are tectonically still active).

(A3) For all cases considered, the effective shear stress \( \tau_{e} \) will generally not assume 0 (counter to Eq. (21c)) as demonstrated earlier in our analysis.

The visualization of stress trajectories and stress intensity around wellbores further reveals the following practical implications:

(B1) The zone of stress perturbation around a wellbore, where the far-field stress magnitude and orientation are affected by the presence and the mud-pressure in the wellbore, does not extend more than 10–15 wellbore radii. The radial stress beyond five wellbore radii distance from the wellbore is equal to 95% the magnitude of the far-field stress in (a) axially symmetric, (b) uni-axial and (c) bi-axial far-field stress regimes.

(B2) Loading of the wellbore by a mud-weight or hydraulic pressure does not enlarge the stress perturbation zone around the wellbore, the effect is constrained to the region outlined in rule (B1).

(B3) So-called balancing of wellbores is impossible when far-field stresses are radial axi-symmetric tensions (such as above active salt domes), as has been illustrated in Section 2.5.

(B4) A reversal of principal stress directions may occur inside the stress perturbation zone for uni-axial and bi-axial far-field stresses (illustrated here in Fig. 11a–d). The isotropic points mark a region around the wellbore where the principal stress reversals occur. This is particularly relevant for fracturing operations [15–18].

(B5) Breakout-stress measurements need a correction for the 90 degree rotation of principal stresses (Fig. 11a–d) that typically occur in the stress-perturbation zones near over-balanced wellbores.

5. Recommendations and conclusions

Risk management of drilling and completion operations demand continuous improvements to wellbore-stability models [35]. This study aims to contribute to the general learning-curve which aims to make drilling operations safer, faster and cheaper [36,37]. The state of stress around mud-load-pressured boreholes affected by a far-field stress can be modeled analytically, commonly based on equations first derived by Gustav Kirsch over a century ago. The principal stress trajectories around wellbores, as they interact with the ambient far-field principal stresses and in response to manipulation of the mud-pressure in the wellbore, have not been mapped systematically before. Our study highlights the importance of attempts to map such trajectories in order to improve wellbore-stability analyses as well as the pre-drilling planning of wellbore-trajectory paths.

Safe-drilling windows in wellbore-stress monitoring-systems need to take into account the effects of the far-field stresses. A universal expression valid for all far-field boundary conditions does not exist: using a generic expression for estimating the stress components on a wellbore (cf. Eqs. (21a)–(21c)) can easily lead to misinterpretations of the stress-concentration factor—too low if the far-field stress is bi-axial plane-stress and too high of the stress is axially symmetric.

The present study systematically reviewed the analytical expressions valid for specific settings and demonstrated that meticulous attention must be paid to the far-field boundary-conditions in order to reveal the appropriate stress-state near the wellbore. Principal stress trajectories around wellbores can assume an unlimited range of patterns, depending upon the boundary conditions (including far-field stresses and wellbore net-pressures), geo-mechanical properties of the rock (incompressible, elastic, or poro-elastic), and thermal conditions. Any additional thermal stresses have been excluded from our analysis, but can be relatively simply accounted for analytically (by a thermal stress function superposition, as these generally are symmetric around the wellbore).

In our study, wellbores were assumed to be aligned with one of the principal stress axes to bring out the salient details of the stress equations. Principal stresses along real wellbores will generally only be aligned with principal stresses at shallow depths due to the free surface condition at zero depth. The stress trajectories around deeper wellbores inclined to the principal stress axes can be derived by incorporating stress-tensor transformation-formulae in our analysis. The widespread practice of using the horizontal total stresses \( \sigma_{H} \) (or \( \sigma_{HMAX} \) or \( \sigma_{HMIN} \)) as inputs for the principal stresses featuring in the wellbore stress equations (as common in nearly all wellbore-stress analysis-tools) is itself a potential source of false conclusions for wellbore stability measures when the principal stresses are no longer aligned with \( \sigma_{H} \) and \( \sigma_{B} \). Practical rules for improved wellbore-stress analyses, based on our review of the analytical expressions currently used, have been formulated in Section 4 (rules A1–A3 and B1–B5).

Acknowledgments

Alboran Energy Strategy Consultants has generally sponsored part of this study.

References