Displacement field potentials for deformation in elastic Media: Theory and application to pressure-loaded boreholes

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ABSTRACT

This study demonstrates how analytical solutions for displacement field potentials of deformation in elastic media can be obtained from known vector field solutions for analog fluid flow problems. The theoretical basis is outlined and a geomechanical application is elaborated. In particular, closed-form solutions for deformation gradients in elastic media are found by transforming velocity field potentials of fluid flow problems, using similarity principles. Once an appropriate displacement gradient potential is identified, solutions for the principal displacements, elastic strains, stress magnitudes and stress trajectories can be computed. An application is included using the displacement gradient due to the internal pressure-loading of single and multiple wellbores. The analytical results give perfect matches with results obtained with an independent discrete element modeling method.

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1. Introduction

Mathematical descriptions of both fluid flow (including flow in porous media) and deformation in elastic media are possible with so-called complex potentials [1,2]. Such descriptions capture in the respective media, the spatial change of fluid velocity (fluid media) and elastic displacements (elastic media). The classical approach of complex analysis for fluid flow splits the complex potential in a stream function (imaginary part) and a potential function (real part). The stream function provides the velocity field and associated velocity gradient tensor for every fluid particle in every location. The potential function gives the pressure field everywhere in the fluid studied. Countless flows can be described by specific stream function solutions [3–7], which all satisfy the Laplace equation. Fig. 1 shows an example [8] of flow past a cylinder with streamlines as velocity tangents obtained from a classical stream function (Fig. 1a) and pressure contours from the potential function (Fig. 1b). Time-of-flight contours given in Fig. 1a based on the analytical model closely match those independently modeled by a physical laboratory experiment [9,10] with marked fluid particles moving around a falling cylinder (Fig. 1c).

Unlike the analytical solutions, the physical laboratory experiment shows streamlines and isochrons affected by experimental "noise," such as wall effects due to finite container size (Region A, Fig. 1c), variable adherence to the fluid boundary due some slip in places of un-intended lubrication (Region B, Fig. 1c), and even non-Newtonian effects, such as wider wake behind the cylinder (Region C, Fig. 1c), due to the high molecular weight of the cross-linked polymer fluid used [11–13]. In short, physical laboratory experiments and computational models, which include both analytical and discrete element solutions methods, are complementary. Analytical model descriptions give exact, closed-form solutions, but are often
limited to isotropic material properties and simple boundary conditions. Advanced discrete element models can handle more complexity, but at expense of longer computation time unless coarse grid and mesh suffice, which give only approximate solutions.

When analytical solutions are available for flow problems they offer high resolution results (meshless, gridless) at low computation cost. For that reason a revival in stream function applications has been advocated in recent efforts to exploit the infinite resolution of closed-form solutions. Examples are flooding studies in hydrocarbon reservoirs [14,15], flow near hydraulically fractured wells [16], and fluid drainage near multi-fractured horizontal wells with fracture hits [17,18]. Separately, complex potentials and associated stress functions have been developed for elastic deformations to map the stress concentrations near internal boundaries [19–21]. Stress functions continue to provide closed-form solutions that can be applied to quantify the elastic response and possible failure of wellbores at high resolution, locating stress trajectories and neutral points of zero deviatoric stress [22,23].

Stream functions and stress functions can both be derived from complex potentials, and use similar tools of complex analysis. One may attempt a transformation of the stream function for a fluid flow system to a stress function to describe a geometrically similar elastic deformation. However, the actual transformation of a stream function to a stress function is less straightforward than alleged, and specific examples are rare if not completely absent in scholarly literature. Goodier [24] argued that by replacing viscosity with the shear modulus and strain rate with strain, the instantaneous, incompressible all-viscous and all-elastic problems are mathematically identical. In order to obtain valid similarity solutions for the kinematic (velocities/displacements) and dynamic (pressures) quantities of such moving fluids and deformed elastic media, the boundary conditions need to be similar (or similarly scalable). In fact, the procedure may be slightly more involved than simply replacing viscosity with shear modulus and strain rate with strain. In an elastic continuum, the deformation in response to a specific external force is instantaneous and results in a specific combination of finite strain and rigid body rotation (neglecting the option of volume change due to compressibility). Provided the finite strain is known everywhere, the stress scaling may require not only the shear modulus but also the Young modulus, because the deformation may involve both pure and simple shear motion. Incompressible, Newtonian fluids, display only shear motion and if scaled for a given (shear) viscosity and assuming steady-state flow, the strain-rate in every point remains time-independent (constant).

This study takes a more fundamental approach and shows how the velocity potential for a suitable flow problem can be manipulated to obtain valid solutions of the displacement potential for elastic problems. Closed-form solutions for advanced
fluid-flow problems may be transformed to solve kinematically similar cases of elastic deformation, where suitable stress functions are not readily available. Our proposed methodology is original and has not been detailed in any prior study. A generic schedule to transform any velocity potential to describe kinematically similar elastic deformation with an analog displacement potential is useful to expand the application of closed-form solutions for practical engineering problems. Such applications are particularly merited when independent stress function formulations do not (yet) exist for the elastic problem. For example, stress functions to analytically solve the stress field between multiple (more than two) pressure-loaded boreholes do not exist (see brief review in Section 4.1). This study shows how solutions for such elastic problems can be readily obtained with our method, given a known velocity potential solutions that is kinematically similar to the elastic problem.

Our approach decouples the elastic displacement field solution from material properties in the sense that various combinations of stress magnitude and elastic moduli may lead to the same displacement field. Fig. 2 shows a generic diagram of basic continuum mechanics domains, distinguishing (1) rigid body displacements, involving translations plus rigid body rotation only, (2) elastic body displacements, involving translations, rigid body rotation and minor strains, and (3) fluid body displacements, involving relatively unconstrained particle movements. This study uses fulfilled compatibility equations as captured in complex potentials of the velocity gradient (fluid flow) and displacement gradient (elastic deformation). A solution for a certain displacement field with compatibility fulfilled can provide stress field solutions by introducing the elastic moduli particular to the case study as governed by a constitutive equation so that energy conservation is accounted for.

This study proceeds as follows. First, we highlight the similarities between tensor operators used to describe elastic displacement and fluid flow, as well as the nearly identical constitutive equations (Section 2). Next, we develop the generic schedule for transposing any velocity potential (Appendix A, B) to a displacement potential in such a way that closed-form solutions can be obtained for geometrically and kinematically similar elastic solutions (Section 3). Subsequently, practical examples are given for pressure-loaded boreholes (Section 4), followed by a brief discussion of application areas (Section 5). Conclusions are given in the final section (Section 6).

2. Kinematical quantities and constitutive relations

2.1. Kinematical quantities

This section first introduces the kinematical quantities due to a certain change in state of a finite material volume. The initial particles of position, $x_j$, for any medium, after loading by certain forces (boundary forces, body forces due to chemical, thermal and/or gravity reactions) may attain new positions, $x_i$, and the coordinate transformation that occurs can be described by the deformation tensor:

$$x_i = F_{ij}x_j \quad (1)$$

For an elastic body, loading by a force will lead to a deformed state and the instantaneous displacement related to the initial state can be described by the displacement gradient tensor ($\nabla U = U_{ij} = \partial u_i / \partial x_j$):

$$u_i = U_{ij}x_j \quad (2)$$

The displacement gradient tensor can be split in its symmetric part (linear strain tensor) and anti-symmetric part (rotation tensor):

$$\nabla U = \frac{1}{2}[\nabla U + \nabla U^T] + \frac{1}{2}[\nabla U - \nabla U^T] \quad (3)$$

The strain and rotation tensors provide practical quantities for later use in the constitutive equation (Section 2.2).
For a fluid body, movement of fluid particles in response to a pressure gradient will occur at a certain rate which can be described by the velocity gradient tensor \( \nabla V = \nabla V = \partial v_i / \partial x_j \):

\[
v_i = V_j \delta_{ij}
\]  

(4)

The velocity gradient tensor can be split in its symmetric part (strain rate tensor) and anti-symmetric part (vorticity tensor):

\[
\nabla V = \frac{1}{2} \left[ \nabla V + \nabla V^T \right] + \frac{1}{2} \left[ \nabla V - \nabla V^T \right]
\]  

(5)

Clearly, the equations describing elastic displacement gradients and fluid velocity gradients employ similar operators. The similarity also extends to their constitutive equations.

2.2. Constitutive relations

The analogy between viscous flow rates solutions and elastic displacements is rooted in the general constitutive relationship which, for a slow viscous flow and linear elastic deformation appear identical (Powers [27], Eqs. (1.782)–(1.790)):

\[
\sigma_{ij} = 2 \mu \varepsilon_{ij} + \lambda \delta_{ij} \varepsilon_{kk}
\]  

(6)

with total stress tensor \( \sigma_{ij} \). In case of the elastic solid, \( \varepsilon_{ij} \), represents the strain tensor, and Eq. (6) is known as Hooke’s Law. Alternatively, \( \varepsilon_{ij} \) is the strain rate \( \dot{e}_{ij} \) in case of the viscous fluid. The scalars in Eq. (6) are \( \mu \) (shear modulus for elastic solid; dynamic shear viscosity for the viscous fluid) and \( \lambda \) (Lamé’s constant for elastic solid; bulk viscosity coefficient for compressible viscous fluid). Kronecker delta ensures that only principal components or taken for the tensor operated on:

\[
\delta_{ij} = \begin{cases} 
0 & i \neq j \\
1 & i = j 
\end{cases}
\]

For a compressible Newtonian fluid, the term \( \lambda \delta_{ij} \varepsilon_{kk} \) replaced by \( \lambda \delta_{ij} \dot{e}_{kk} \) in Eq. (6) represents the volumetric strain with pressure storage equal to the bulk viscosity coefficient, \( \lambda \), times the sum of the principal elongation rates, \( \dot{e}_{kk} \). For an incompressible Newtonian fluid the term \( \lambda \delta_{ij} \dot{e}_{kk} \) can be replaced by \( P \delta_{ij} \), with \( P \) denoting the pressure scalar. Similarly, in the elastic medium, the volumetric strain is due to the elongation, with \( \lambda \) representing the proportionality of the volumetric strain to the component of the total stress used to achieve the volumetric strain.

The constitutive relationship between the applied stress and elastic deformation in terms of the symmetric part of the displacement gradient tensor is:

\[
\sigma_{ij} = \mu \left( \frac{\partial u_k}{\partial y} + \frac{\partial u_k}{\partial x} \right) + \lambda \delta_{ij} \nabla \cdot \vec{u}
\]  

(7)

The first term accounts for distortion that leads to deviatoric stress accumulations controlled by the shear modulus. The second term represents the pressure storage due to the isochoric deformation controlled by Lamé’s constant, alternatively termed here the pressure storage coefficient. A similar expression can be formulated for fluid flow, using the velocity gradient tensor instead of the displacement gradient tensor.

3. Elastic deformation and displacement potential

3.1. Displacement potential

The generic method of abstracting, from the stream function for a given flow, the velocity gradient tensor, is given in Appendix A, which includes logical steps to subsequently obtain the spatial variation in the principal strain rate magnitudes and orientation of the principal strain rate axes. Coupled with a constitutive equation to scale the stream function and strain rate tensor with a particular pressure gradient solves for the absolute flow rates. Appendix B explains how complex potential calculus can be applied to analytical stream functions and includes a specific example on (1) how the velocity gradient tensor elements are computed for multiple point source flows, and (2) scaled for dimensional rates of those sources.

Instead of having velocity gradients like in the fluid flow case (Appendices A and B), we have displacement and displacement gradients in the elastic deformation case. For viscous problems, we have shown the viscosity can be scaled out to give the streamlines independent of fluid properties (Appendix A1), except for a requirement of being incompressible and having a uniform, linear viscosity. When the boundary condition is a certain volumetric flux rate (and not a pressure differential), the solution of the stream function is valid for a fitting range of viscosities and strain rates [37]. For elastic problems, a similar approach is possible, scaling out the deforming boundary stresses, and instead specifying simply the strains imposed at the boundaries (see below).

Suppose we understand the boundary conditions and have established the initial conditions at, and geometry of, any physical boundaries of an elastic deformation process and want to transform an appropriate stream function to solve the corresponding elastic problem. In the present study, we consider an infinite elastic medium with multiple elastic displacement...
sources (due to local pressure loading), without any far-field deformation. The imposed elastic displacement field solution initially scales out material properties. The advantage of this approach is that the complex potential of the 2D displacement field will be valid for various combinations of stress magnitude and elastic moduli leading to the same displacement field.

In our specific example, the viscous velocity potential for multiple flow sources (Appendix B) can be transformed to an elastic displacement potential of a complex plane with multiple displacement sources:

$$U(z) = \sum_{s=1}^{n} \frac{\epsilon_s}{(z - z_s)}$$

(8)

The imposed areal displacements (e.g., change in the wellbore radius due to an internal net pressure), treated as point sources (transposing strength $m_i/z\pi$ by radial displacement $\pi r_s^2/2\pi$) are scaled by $\epsilon_s = \frac{z_s^2}{2\pi}$. Displacements occurring in the elastic continuum due to an initial areal strain $\epsilon_s$ [m²] at each wellbore can be quantified throughout the continuum as follows.

For the multiple wellbore problem the displacement vectors in x and y direction ($u_x$ and $u_y$) are completely described by $U(z) = u_x - iu_y$. The displacement in the x and y directions are obtained in the following derivation:

$$U(z) = \sum_{s=1}^{n} \frac{\epsilon_s}{(z - z_s)} = \sum_{s=1}^{n} \frac{\epsilon_s}{(x + iy)(x_s + iy_s)} = \sum_{s=1}^{n} \frac{\epsilon_s}{(x - x_s) + i(y - y_s)}$$

$$= \sum_{s=1}^{n} \frac{\epsilon_s}{(x - x_s) + i(y - y_s)} \cdot \left\{ \frac{(x - x_s) - i(y - y_s)}{(x - x_s)^2 + (y - y_s)^2} \right\}$$

$$= \sum_{s=1}^{n} \frac{\epsilon_s(x - x_s)}{(x - x_s)^2 + (y - y_s)^2} - i \sum_{s=1}^{n} \frac{\epsilon_s(y - y_s)}{(x - x_s)^2 + (y - y_s)^2} = u_x - iu_y.$$  

(9)

Hence we find:

$$u_x = \sum_{s=1}^{n} \frac{\epsilon_s(x - x_s)}{(x - x_s)^2 + (y - y_s)^2}$$

(10a)

$$u_y = \sum_{s=1}^{n} \frac{\epsilon_s(y - y_s)}{(x - x_s)^2 + (y - y_s)^2}$$

(10b)

These are the two displacement vectors ($u_x$ and $u_y$) in x and y direction throughout the elastic continuum, which thus describe the full displacement field. Such a displacement field maintains compatibility (Fig. 2) and can provide specific stress field solutions by later introducing the elastic moduli particular to the case study, as governed by a constitutive equation.

3.2. Displacement gradient tensor

The above displacement field is a valid solution of the elastic displacement potential $\zeta = U(z)$, which must fulfill the compatibility and continuity equations, which are automatically ensured when $\zeta$ is a valid solution of the biharmonic equation:

$$\nabla^4 \zeta = \frac{\partial^4 \zeta}{\partial x^4} + 2\frac{\partial^4 \zeta}{\partial x^2 \partial y^2} + \frac{\partial^4 \zeta}{\partial y^4} = 0$$

(11)

Above formulation assumes there are no body forces, only forces exerted at any internal and external boundaries are considered. The generic displacement gradient tensor $\nabla U$ can be expressed in terms of the elastic displacement potential $\zeta$:

$$\nabla U = \begin{bmatrix} \frac{\partial^2 \zeta}{\partial x \partial y} - \frac{\partial^2 \zeta}{\partial y \partial x} \\ \frac{\partial^2 \zeta}{\partial x^2} - \frac{\partial^2 \zeta}{\partial y^2} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_x}{\partial y} & \frac{\partial u_y}{\partial x} \\ \frac{\partial u_x}{\partial x} & \frac{\partial u_y}{\partial y} \end{bmatrix}$$

(12)

Differentiation of the displacement potential $U(z)$ with respect to x and y using the specific displacement vectors of Eqs. (10a) and (10b) yields the displacement gradient tensor for multiple wellbores:

$$\nabla U = \begin{bmatrix} \sum_{s=1}^{n} \frac{\epsilon_s}{(x_s - x)^2 + (y_s - y)^2} \\ \sum_{s=1}^{n} \frac{\epsilon_s}{(x_s - x)^2 + (y_s - y)^2} \end{bmatrix} \begin{bmatrix} \sum_{s=1}^{n} \frac{-2(x_s - x) \cdot (y - y_s)}{(x_s - x)^2 + (y_s - y)^2} \\ \sum_{s=1}^{n} \frac{-2(x_s - x) \cdot (y - y_s)}{(x_s - x)^2 + (y_s - y)^2} \end{bmatrix} = \begin{bmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{bmatrix}.$$  

(13)
The displacement gradient tensor $\nabla U$ can be decomposed in the symmetric (strain tensor) and anti-symmetric (rotation tensor) parts:

$$
\nabla U = \begin{bmatrix}
\frac{\partial^2 \xi}{\partial x \partial y} & \frac{1}{2} \left( \frac{\partial^2 \xi}{\partial y^2} - \frac{\partial^2 \xi}{\partial x^2} \right) \\
\frac{1}{2} \left( \frac{\partial^2 \xi}{\partial y^2} - \frac{\partial^2 \xi}{\partial x^2} \right) & -\frac{\partial^2 \xi}{\partial y \partial x}
\end{bmatrix} + \begin{bmatrix}
0 & \frac{1}{2} \left( \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial x^2} \right) \\
-\frac{1}{2} \left( \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial x^2} \right) & 0
\end{bmatrix}
$$

The first matrix is the strain tensor $\Gamma$ and the second matrix is the rigid rotation tensor $\Omega$.

$$
\nabla U = \begin{bmatrix}
u_{xx} & \frac{1}{2} \left( u_{xy} + u_{yx} \right) \\
\frac{1}{2} \left( u_{xy} + u_{yx} \right) & u_{yy}
\end{bmatrix} + \begin{bmatrix}
0 & \frac{1}{2} \left( u_{xy} - u_{yx} \right) \\
\frac{1}{2} \left( u_{xy} - u_{yx} \right) & 0
\end{bmatrix}.
$$

In other notation $\nabla U = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} \\ \epsilon_{xy} & \epsilon_{yy} \end{bmatrix} + \begin{bmatrix} 0 & \vartheta \\ -\vartheta & 0 \end{bmatrix}$, where the first matrix is the strain tensor $\Gamma$ and the second matrix is the rigid rotation tensor $\Omega$.

3.3. Computing the strain tensor and principal strains

Substituting displacement gradient tensor elements into the strain tensor gives:

$$
\epsilon_{xx} = \sum_{s=1}^{n} \epsilon_s \frac{-(x-x_s)^2 + (y-y_s)^2}{((x-x_s)^2 + (y-y_s)^2)^2} 
$$

$$
\epsilon_{yy} = \sum_{s=1}^{n} \epsilon_s \frac{(x-x_s)^2 - (y-y_s)^2}{((x-x_s)^2 + (y-y_s)^2)^2}
$$

$$
\epsilon_{xy} = \sum_{s=1}^{n} \epsilon_s \frac{-2(x-x_s) \cdot (y-y_s)}{((x-x_s)^2 + (y-y_s)^2)^2}
$$

The principal strain magnitude, the maximum shear strain and isoclines of the principal strain axes can now be obtained as follows:

Principal strain magnitude:

$$
\epsilon_1, \epsilon_2 = \frac{\epsilon_{xx} + \epsilon_{yy}}{2} \pm \sqrt{\left( \frac{\epsilon_{xx} - \epsilon_{yy}}{2} \right)^2 + \epsilon_{xy}^2}
$$

Shear strain magnitude:

$$
\epsilon_{xy, \text{max}} = \sqrt{\left( \frac{\epsilon_{xx} - \epsilon_{yy}}{2} \right)^2 + \epsilon_{xy}^2}
$$

Orientation of principal strain axes:

$$
\tan(2\theta_s) = \frac{2\epsilon_{xy}}{\epsilon_{xx} - \epsilon_{yy}}
$$

4. Geomechanical application

In our application below, we first outline state-of-the-art studies of borehole geomechanics and argue why our current analytical approach is merited (Section 4.1). Next, a numerical benchmark is set (Section 4.2), followed by our analytical solution method (Section 4.3). An additional example is given for boreholes with variable pressure loading (Section 4.4).

4.1. Prior work on borehole geomechanics

Borehole problems in geological and geotechnical applications are typically studied analytically using the so-called Kirsch equations [39,41-46] derived from the classical expressions of Kirsch [47], which in turn used solutions from the contemporary mechanical engineering textbooks by Bach [48] and Föppl [49]. The Kirsch equations assume homogeneous elastic, linear response and applications include modeling the state of stress around geological intrusions (of magma, salt or mud [22], and in geotechnical studies of wellsbores, shafts, tunnels, and man-made cylindrical storage caverns such as constructed by solution mining in salt domes [23]. The Kirsch equations have found additional applications in mechanical engineering of stressed elastic plates with holes [19-21,38,39,50,51]. Stress patterns around boreholes can also be studied in finite element
models, typically involving poro-elastic effects [52,53], but the analytical Kirsch equations provide a powerful alternative for stress quantification around the wellbore.

Although very practical for use on single boreholes, the Kirsch equations do not allow reformulation in terms of complex potentials to enable studies of the interference of stress and elastic displacement due to simultaneous perforation by a large number of multiple boreholes. Analytical solution for the interaction of a pair of circular holes in an infinite elastic (and visco-elastic) medium have been formulated in several studies [54–58]. A finite-element model of two neighboring circular holes quantified the stress concentration due to the internal pressure of a non-explosive expansion material used in mining applications [59]. Analytical solutions for the stress interference between a larger number of multiple, pressure-loaded circular holes have not been derived before - a gap in concurrent literature which we intend to fill with our present analysis. Applications of the theory developed in our paper to single and multiple borehole(s) are illustrated below.

4.2. Numerical benchmark pressure-loaded drill holes

In order to be able to validate our analytical code and solutions, we first used Abaqus, a numerical discrete element model, to plot displacement fields for simple cases of two closely spaced boreholes. The sign convention followed here is that extension is a positive strain and shortening is a negative strain. Likewise, tensile stress is considered positive and compressive deviatoric stress is negative, in accordance with the common conventions in generic continuum mechanics, but opposite to the sign convention commonly used in geomechanics literature.

The distance between the centers of the first pair of drill holes with identical radii of 0.25 is 3 non-dimensional length units (or 12 times the radius, Fig. 3a). A second well pair uses the same spacing, but has smaller radii of 0.05, so that the inter-well distance is 60 times the well radius (Fig. 3b). Decreasing radii were used to examine any discernable change in the displacement field. Comparison of the contour patterns of Figs. 4a and b suggests the displacement contour patterns (total displacement and its Cartesian vector components) were barely affected by decreases of the wellbore radii. Figs. 4a and b show similar contour patterns, which only differ in absolute scaling. We conclude that when the ratio of the wellbore radii and spacing remains small (say less than 1/10) we do not see any effect of the finite wellbore radii on the displacement contours, which justifies the approximation of wellbores by singularities in our analytical models (Section 4.3).

However, we did notice the strains in Figs. 4a,b were affected by boundary effects due to the proximity of the external no-slip box boundaries. We subsequently constructed two more models (radii 0.1 and 0.02, respectively) free from external boundary constraints (infinite domain). The results of Fig. 5a,b show that the displacement patterns differ from those for the bounded domain cases in Fig. 4a,b. For benchmarking our analytical results with the same wellbore separation, we must use the unbounded case of Fig. 5a,b.

4.3. Analytical results: base cases

For our analytical models, three different cases are considered (Fig. 6): (1) a single borehole located at the origin, (2) two boreholes located on the X-axis and at x = ± 1, and (3) seven boreholes, one located at origin and the others occurring at the corners of a hexagon centered on the origin with radius one. In the following, we present the results of our modeling schedule when applied to a homogeneous, isotropic, linear elastic continuum penetrated by pressure-loaded, cylindrical boreholes, oriented mutually parallel and perpendicular to the complex plane of the modeling space. Our present analysis represents the borehole by singularities. The boreholes are pseudo-discontinuities in the sense that displacement gradients can be initiated from the singularity but no finite hole exists. The approximation by a singularity means no shear stress/strain shall exist on the wellbore, which is satisfied when no far-field stress exists. The principal stresses and strains and strains will be radial and tangential in the absence of a far-field stress.
Fig. 4. Abaqus model results. Plots of displacement field (total magnitude, and Cartesian components) around two pressurized boreholes in bounded domain. Borehole radii in the bottom row are 1/5th of those in the top row.

Fig. 5. Abaqus model results. Plots of displacement field (total magnitude, and Cartesian components) around two pressurized boreholes in unbounded domain. Borehole radii in the bottom row are 1/5th of those in the top row.

Fig. 7 shows the total displacement field and Cartesian components computed based on our analytical displacement potential of Eq. (8) for each borehole configuration shown in Fig. 6. All plots were produced using a simple matlab code. The solutions for the displacements of Fig. 7 follow from the spatial velocity field given by the complex displacement potential of Eq. (8), with displacements solved for in Eqs. (10a,b). Comparison of the 2-hole case in Fig. 7 (middle column) and Fig. 5 shows excellent matches between the analytical and numerical solutions for the total displacement field, as well as for the Cartesian components.
The displacement gradient tensor [Eq. (13)] is used to solve for the strain magnitudes and orientations (Fig. 8). The tensorial strain elements follow from Eq. (16a), the principal stress magnitude from Eq. (17) and the trajectories for the compressive principal strain axis from Eq. (19). In all cases, the strength, $\varepsilon_s$, of the elastic deformation at the boreholes is set uniformly to $10^{-4}$ (non-dimensionalized via division by the square of the length unit used), elastic modulus $E_Y = 10^3$ (non-dimensionalized via division with the pressure unit used), and Poisson ratio $\nu = 0.3$. With $E_Y$ and $\nu$ fixed, recall the following relationships exist between the elastic constants:

$$\gamma = E_Y / [2(1 + \nu)]$$

(20a)

$$\lambda = 2\gamma \nu / (1 - 2\nu)$$

(20b)

Eq. (20a) is used to compute $\gamma$ from the inputs for $E_Y$ and $\nu$. Any fractional change in volume due to a confining pressure $P$ [with $P = (\sigma_1 + \sigma_2 + \sigma_3) / 3$] is given by $P/K$, with bulk modulus $K$, which is also fixed by for $E_Y$ and $\nu$:

$$K = E_Y / [3(1 - 2\nu)]$$

(21a)

$$\nu = (3K - 2\gamma) / [2(3K + \gamma)]$$

(21b)

The actual magnitude of the stress tensor $\sigma_{ij}$ solutions in the vicinity of the boreholes depends on the magnitudes of their individual stretches, substituted as the initial condition at the borehole singularity locations $(x_s, y_s)$ in the deforming
Fig. 8. Analytical model results. **Top Row:** Tensile principal strain is denoted as $\varepsilon_1$ for each case, and all principal strains are coincident with the $xy$ plane, perpendicular to the boreholes, with $x$ horizontal and $y$ vertical marked with non-dimensional length scales. **Middle Row:** Tensor strain element $\varepsilon_{xx}$ for single boreholes and arrangements of 2 and 7 boreholes. **Bottom Row:** Strain trajectories for the compressive principal strain axis $\varepsilon_2$ for each case. The 2-hole case has 5 locations along the central vertical line acting as strain saddle points, which are alternating maxima and minima. The 7-hole case has 30 such locations.

A medium described by the displacement potential. The resulting gradients of displacement throughout the elastic medium are determined not only by the boundary condition but also vary with the engineering constants as given in Eqs. (20a,b). Reversely, all elastic stresses vanish if there is no displacement gradient, which occurs when there is a uniform (rigid body) displacement.

For a general 3D deformation of an isotropic elastic continuum, including both pure and simple shear, the stress tensor elements can be computed from the strain tensor elements, expanding Eq. (6):

\[
\sigma_{xx} = (2\gamma + \lambda)\varepsilon_{xx} + \lambda\varepsilon_{yy} + \lambda\varepsilon_{zz}
\]  
(22a)

\[
\sigma_{yy} = \lambda\varepsilon_{xx} + (2\gamma + \lambda)\varepsilon_{yy} + \lambda\varepsilon_{zz}
\]  
(22b)

\[
\sigma_{zz} = \lambda\varepsilon_{xx} + \lambda\varepsilon_{yy} + (2\gamma + \lambda)\varepsilon_{zz}
\]  
(22c)

\[
\sigma_{xy} = 2\gamma\varepsilon_{xy}
\]  
(22d)

\[
\sigma_{xz} = 2\gamma\varepsilon_{xz}
\]  
(22e)
\[ \sigma_{yz} = 2\nu\varepsilon_{yz} \]  

(22f)

The magnitudes of the principal stresses, \( \sigma_1 \) and \( \sigma_2 \) everywhere in the plane of observation can now be obtained by applying the standard expressions:

\[ \sigma_1 = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) + \left[ \sigma_{xy}^2 + \frac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 \right]^{\frac{1}{2}} \]  

(23a)

\[ \sigma_2 = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) - \left[ \sigma_{xy}^2 + \frac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 \right]^{\frac{1}{2}} \]  

(23b)

The strain tensor solutions (Fig. 8) follow from the displacement gradient tensor of Eq. (8) using Eqs. (16a–c) and the magnitude and orientation of the principal strains are given by Eqs. (17) and (19). The maximum shear strain is given by Eq. (18). The strain tensor solutions allow for determination of the stress tensor using Eqs. (22a–f) and principal stress magnitudes given in Eq. (23a,b) act within the plane of deformation. The principal stresses are the maximum and minimum normal stresses that may act on certain material planes through the point of observation [39]. The compressive strain trajectories (Fig. 8, bottom row) indicate the most likely direction of tensile failure due to dilation being facilitated best in a direction normal to the compressive strain trajectories. The principal stress and strain trajectories are identical at all times for the isotropic elastic continuum considered in our study. The principal compressive stress trajectories (Fig. 8, bottom row) predict the curvi-linears for tensile fracture propagation, which for the 7 borehole case will mostly follow perfect straight radials going outward from the central wellbore via the peripheral boreholes. Although the three key parameters \((\varepsilon_s, E_Y, \nu)\) are kept the same for all boreholes in the examples of Figs. 7–9, there is no limitation (see Section 4.5) to varying the strengths \(\varepsilon_s\) of each borehole independently \((E_Y, \nu)\) should be kept constant.

Having the ratio \(\sigma_{xx}/\sigma_{xy}\) solved for in Eqs. (22a) and (22b) allows for the determination of the principal stress axes orientation ([40], Fig. 5 using Mohr expressions) in every location of the elastic displacement field:

\[ \frac{\sigma_{xx}}{\sigma_{xy}} = \frac{\cos(2\xi)}{\sin(2\xi)} = \tan^{-1}(2\xi) \]  

(24)

The principal stress magnitudes around the wellbores can be contoured (Fig. 9) using a constitutive equation, as detailed in Eqs. (22a–sf). Obviously, the stress intensifies in the central region occupied by the perforation cluster. The singularity approach leads to infinitely large stress concentrations in the singularities. Such unrealistic, infinite values can be avoided by excluding the singularity outputs and scaling the stress field by know values at a certain distance away from the singularity equivalent to a certain finite wellbore radius.

---

**Fig. 9.** Analytical model results. Top Row: Magnitude of the largest tensile principal stress denoted as \(\sigma_1\), Bottom Row: Magnitude of compressive principal stress \(\sigma_2\), for each case.
A prior study [44] has shown that stress perturbations of a native, tectonic stress field by penetration of a wellbore only occur within a region up to 10 radii away from the wellbore. However, in the absence of a tectonic stress, as assumed in our present study, wellbores may interfere at greater distances. For example, the stress patterns of Figs. 9 apply to well spacing in non-dimensional units and can be multiplied by any practical length unit of choice (say 1 m, 10 m, 100 m, or 1 km). The implication is that stress trajectories will curve over large regions, the absolute magnitude of the stress being determined by the displacements at the wellbore and the compliances.

4.4. Back-calculating the strain tensor from the stress tensor

For completeness, the general 3D strain tensor formulation accounting for deformation due to both pure and simple shear components becomes possible when expressing the infinitesimal strain in a relationship containing the Poisson ratio, \( \nu \), and Young’s modulus, \( E \):

\[
\varepsilon_{ij} = \frac{(1 + \nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}}{E} \tag{25}
\]

The magnitude of the strain tensor elements can be obtained from the stress tensor elements as follows:

\[
\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right] \tag{26a}
\]

\[
\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right] \tag{26b}
\]

\[
\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{yy} + \sigma_{xx})\right] \tag{26c}
\]

\[
\varepsilon_{xy} = \frac{\sigma_{xy}}{2\nu} \tag{26d}
\]

\[
\varepsilon_{xz} = \frac{\sigma_{xz}}{2\nu} \tag{26e}
\]

\[
\varepsilon_{yz} = \frac{\sigma_{yz}}{2\nu} \tag{26f}
\]

For a general 2D plane deformation with plane strain the only non-zero strain tensor elements are: \( \varepsilon_{xx} \), \( \varepsilon_{yy} \), and \( \varepsilon_{xy} \), which allows for the determination of the 2D stress tensor. Note that not necessarily \( \sigma_{zz} \neq 0 \) for plane strain cases where \( \sigma_{kk} \) is not the principal stress; although \( \varepsilon_{zz} = 0 \) for plane strain.

4.5. Analytical results: variable borehole displacements

The examples in the previous sections were confined to multiple wellbores, all having equal initial displacements at the point sources. The analytical expression of Eq. (8) allows for individual initial displacement assignment \( \varepsilon \), for each wellbore \( (s = 1, \ldots, n) \). The case modeled in Fig. 10 uses the same wellbore configuration as in Fig. 3c, but now the central well has an internal net displacement 5 times larger than for the peripheral wellbores. The displacements and strains near the peripheral wellbores are almost negligible as compared to that of the central wellbore (Fig. 11), which began to dominate the elastic displacement field. The corresponding principal stress magnitudes and orientation are also given in Fig. 11.

5. Discussion

Our complex analysis method (CAM) and results given for pressure-loaded boreholes may provide useful insight for geotechnical applications such as in the design phase of subparallel shafts and tunnels. The locations of high stress concentrations can be quickly evaluated and when coupled with failure criteria (not elaborated here) can be used in a stability analysis to avoid the occurrence of either spalling, due to tension failure, or breakout, due to shear failure. Although discrete element and finite difference based models may be constructed for detailed designs, the CAM-based solutions provide infinite resolution, albeit only up to the scale where continuum mechanics is valid. All our models above are developed based on that assumption. Important locations, such as the neutral points (zero strain, no deviatoric stress) can be readily identified using closed-form solutions. For example, 30 neutral points can be seen in the strain trajectory plots for 7 boreholes (Fig. 8, right column) and 5 for 2 boreholes (Fig. 8, middle column), which are the locations where the contours for the principal strain axes cross.

Possible applications of our method in mining operations include the determination of optimum drilling patterns and wellbore spacing for crack-reaming of rock between boreholes pressured by non-explosive expansion material [59]. Another relevant area of application may be the design of well spacing for cavern creation by solution mining in rock salt domes, which are abundantly used for storage of hydrocarbon energy sources, such as natural gas, petroleum, and ethylene [60–66].
and occasionally for nuclear waste, both in the US and abroad. In contrast to the radial compression occurring when expansive material is introduced for pressure-loading of wellbores [59], wellbores in salt will experience an underpressure and the induced radial tension may lead to concentric tension failure on short-time scales [23], whereas visco-elastic properties of rock salt may lead to crystalline creep on longer time-scales [67]. Our model method can quantify stresses for any constellation of sub-parallel boreholes, and allows for optimization of the well patterns, both for speed of cavern completion (observing safety margins for stable construction) and for economic performance of such construction projects.

Another significant field of possible application of our methods is in two types of operations related to oil and gas extraction. A first petroleum business application is in regular drilling operations, where wellbore stability analysis must ensure drilling and pressure management of the wellbore stays within the safe drilling margins [46]. Increasingly, clusters of wells are drilled from single well pads (onshore) and from single platforms (offshore) in so-called extended reach drilling with close attention to collision damage (Fig. 12). Our method may provide support to avoid wellbore integrity problems due to overly tight spacing of wells in such multi-well drilling operations. A second oil industry application is in hydraulic fracture treatments used to stimulate well productivity in unconventional oil and gas occurrences in shale fields. The hydraulic fractures typically propagate in the direction of the maximum principal stress trajectories, which our models can readily identify for any constellation of closely spaced wellbores. If open hole completions are assumed, hydraulic fracturing in a
stimulated section of the wellbore will begin where the critical stress limit is reached, which typically is in the regions where the tensile stress reaches the tensile failure strength of the rock. Our algorithms allow for quick generation of plots and evaluation of the principal stress trajectories and stress magnitude contours for both single wells and multiple, simultaneously stimulated, wells. Such plots are helpful to identify where failure is likely being initiated first, in order to better plan, control and guide the propagation of such fractures. Improved control of fracture placement and prediction of the likely direction of fracture propagation are particularly merited during the design phase of any fracture treatment job, to ensure costly drilling and stimulation phases occur as planned.

A final area of application may be in geological studies of magma pipes, occurring in subvolcanic complexes. Multiple pipes may branch upward in subparallel, subvertical arrangements, emanating from a central magma chamber. Fractures may form between such magma pipes and lead to intrusions of magma to create felsic dikes, which may become exhumed over geologic time due to erosion and tectonic uplift, such as the Spanish Peaks in New Mexico [68,69]. Stress models based on our methodology may be helpful for understanding the formation of magmatic dike patterns on Earth and its neighboring planets (Mars and Venus display abundant dike patterns). The creation of new magmatic dikes in regions with active volcanism located to major population centers would pose a natural hazard; we suggest our model may provide a practical tool for predictive studies of subsurface failure zones.

6. Conclusions

Our study models linear elastic deformations with complex analysis methods (CAM). In particular, CAM-solutions for the velocity gradient tensor for certain fluid flow problems can be transposed to obtain the displacement gradient tensor for corresponding elastic deformation problems. The theory is first outlined in our study and an example is given for geomechanic applications involving single and multiple borehole(s). The analytical results provide and excellent match with results obtained using an independent discrete element solution method. The elastic deformation imposed by each deformation attribute (e.g., multiple point source displacements) determines the total deformation pattern, which develops instantaneously. In the examples discussed, drill holes are approximated by point sources and poro-elastic effects remain negligible when assuming instantaneous responses to the imposed deformation. The principal strain and principal stress trajectories and contours for the magnitudes of principal stress and strain are obtained by quantifying the displacement gradient due to
borehole pressure and compliances using analytical methods based on complex analysis. Additionally, there is a limit to the elastic deformation the continuum can sustain. When the stress reaches the elastic limit, discrete failure zones will develop and the continuum locally looses cohesion. Our Matlab code is available for further study.

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Appendix A: Generic stream function solutions for flow problems

A.1. Valid stream functions

The Navier-Stokes equation for an incompressible fluid (constant density $\rho$) and Newtonian dynamic viscosity ($\mu$) can be expressed in terms of the Jacobian determinant using a stream function $\psi$:

$$\frac{\partial}{\partial t} \nabla^2 \psi + \frac{\partial (\psi, \nabla^2 \psi)}{\partial (x, y)} - \nu \nabla^4 \psi = 0$$ (A1)

This assumes there are no body forces other than those due to a pressure gradient. The kinematic viscosity $\nu = \mu/\rho$, accounts for constant material properties of the fluid continuum with density, $\rho$, and dynamic viscosity, $\mu$. The first two terms in Eq. (A1) are inertia terms, which vanish in creeping flow and any Darcy flow, so that the Jacobian flow descriptor becomes:

$$\nu \nabla^4 \psi = 0$$ (A2a)

When fluid viscosity is scaled out, Eq. (A2a) can be further simplified to:

$$\nabla^4 \psi = 0$$ (A2b)

The description may still apply to viscous flow, and the viscosity may have any possible value, but remains constant throughout a given flow space. Scaling the viscosity out provides a concise, inviscous flow description, but re-introducing a case-specific viscosity consequent to an appropriate constitutive equation allows for scaling of time-of-flight contours and stress magnitudes [25,26].

A.2. Generic velocity gradient tensor

The velocity gradient tensor can be readily derived from the velocity matrix. This is useful in order to find the strain rate. Tensor theory states that any second rank tensor can be decomposed into a so-called symmetric and anti-symmetric tensor [27]. These two tensor operators, when applied to the velocity gradient tensor, are known as the strain rate tensor and the rigid rotation rate tensor (or “vorticity” or “curl”), respectively. The velocity gradient tensor can be expressed in terms of the stream function:

$$\nabla V = \begin{bmatrix} \frac{\partial^2 \psi}{\partial x \partial y} & \frac{\partial^2 \psi}{\partial y^2} & \frac{\partial^2 \psi}{\partial x^2} \\ \frac{\partial v_x}{\partial x} & \frac{\partial v_y}{\partial y} & \frac{\partial v_z}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_y}{\partial y} & \frac{\partial v_z}{\partial z} \\ \frac{\partial v_x}{\partial y} & \frac{\partial v_y}{\partial x} & \frac{\partial v_z}{\partial z} \end{bmatrix}. \quad \text{(A3)}$$

The velocity gradient tensor $\nabla V$ can be decomposed as follows:

$$\nabla V = \frac{1}{2} [\nabla V + \nabla V^T] + \frac{1}{2} [\nabla V - \nabla V^T] \quad \text{(A4)}$$

The decomposition in the symmetric and the anti-symmetric part is:

$$\nabla V = \begin{bmatrix} \frac{\partial^2 \psi}{\partial x \partial y} & \frac{1}{2} \left( \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2} \right) \\ \frac{1}{2} \left( \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2} \right) & \frac{\partial^2 \psi}{\partial y \partial x} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} \left( \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2} \right) \\ -\frac{1}{2} \left( \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2} \right) & 0 \end{bmatrix}. \quad \text{(A5)}$$

The first matrix is the strain rate tensor $E$ and the second matrix is the rigid rotation rate tensor $W$ (Malvern, 1969). In other notation $\nabla V = \begin{bmatrix} \dot{E}_{xx} & \dot{E}_{xy} \\ \dot{E}_{yx} & \dot{E}_{yy} \end{bmatrix} + \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$, where the first matrix is the strain rate tensor $E$ and the second matrix is the rigid rotation rate tensor $W$. 

A.3 Obtaining the principal strain rates

The principal strain rates $\dot{e}_1, \dot{e}_2$ can be expressed in terms of the strain rate tensor’s unique components in an arbitrary coordinate system (Cartesian assumed here, with x,y plane coincident with the plane in which $\dot{e}_1, \dot{e}_2$ act):

$$
\dot{e}_1, \dot{e}_2 = \frac{\dot{e}_{xx} + \dot{e}_{yy}}{2} \pm \sqrt{\left(\frac{\dot{e}_{xx} - \dot{e}_{yy}}{2}\right)^2 + \dot{e}_{xy}^2}.
$$

(A6)

By the sign convention in geomechanics, $\dot{e}_1$ (mostly compression) is taken positive and $\dot{e}_2$ (mostly extension) is taken negative, opposite to the sign convention used in mechanical engineering. In our study we use the mechanical engineering sign convention. This study includes a geo-mechanic application (Section 4; which also assigns a positive sign for extensional elastic strains).

The maximum shear strain rate is:

$$
\dot{e}_{xy,max} = \sqrt{\left(\frac{\dot{e}_{xx} - \dot{e}_{yy}}{2}\right)^2 + \dot{e}_{xy}^2}
$$

(A7)

and maximum shear strain rate angles are respectively given by:

$$
\tan (2\theta_s) = \frac{2\dot{e}_{xy}}{\dot{e}_{xx} - \dot{e}_{yy}}, \theta_s = \theta \pm 45^\circ.
$$

(A8)

The principal strain rates expressed in terms of the stream function are:

$$
\dot{e}_1, \dot{e}_2 = \frac{1}{2} \left( \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} \right) \pm \frac{1}{2} \sqrt{\left( \frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial y \partial x} \right)^2 + \left( \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2} \right)^2}
$$

(A9)

The maximum shear strain rate is:

$$
\dot{e}_{xy,max} = \frac{1}{2} \sqrt{\left( \frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial y \partial x} \right)^2 + \left( \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2} \right)^2}
$$

(A10)

The maximum shear strain rate angles are respectively given by:

$$
\tan (2\theta_s) = \left( \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2} \right) \left( \frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial y \partial x} \right), \theta_s = \theta \pm 45^\circ.
$$

(A11)

The above equations allow the visualization of the velocity field, strain rates and the shear strain rate angle.

A.4 Scaling a stream function and strain rate tensor solutions for stress

The total stress tensor for incompressible viscous flow is:

$$
\sigma_{ij} = -P\delta_{ij} + \tau_{ij}
$$

(A12)

The static pressure $P$, due to uniform compression equals the average stress $\sigma_{kk}/3$, and the deviatoric or viscous stress tensor $\tau_{ij}$ accounts for the viscous flow resistance of the fluid continuum, where the Kronecker delta $\delta_{ij}=1$ if $i=j$, and $\delta_{ij}=0$ otherwise.

The viscous stress $\tau_{ij}$ depends on gradients of velocity and vanishes if there is no velocity gradient (uniform flow). The original flow description can now be dimensionalized by adopting Eq. (A2a), valid for a fluid continuum with a specific viscosity. The preferred viscosity measure to express shear resistance is the dynamic viscosity $\mu$:

$$
\tau_{ij} = 2\mu \left[ \frac{\partial^2 \psi}{\partial x \partial y} \right] \left[ \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2} \right]
$$

(A13)

The deviatoric stresses $\tau_{xx}$ for any point in the flow follows from the principal strain rate in a constitutive relationship:

$$
\tau_{ij} = 2\mu \dot{e}_{ij}
$$

(A14)

The dynamic viscosity $\mu$ accounts for viscous resistance of the fluid continuum. Applications of potential theory are not limited to inviscid flow and may be applied to incompressible viscous flow without adaptation [28,29]. Application to
natural examples scaled for viscosity are given in prior studies for lava extrusion sheets [25], creep of submarine salt sheets in the Gulf of Mexico [30,31], and gravity spreading of mud flows, glaciers and namakiers [32]. The viscosity in such flows can be scaled as long as the system modeled can be simulated without incurring any net vorticity.

Flow descriptions by potential functions are valid for incompressible fluids subjected to irrotational flow. It is therefore often assumed that potential flow descriptions and the implicit stream function descriptions require a fluid to be inviscous. Joseph [28] has explained in detail why this interpretation is incorrect: irrotational flow is a property of the flow and viscosity is a property of the material; irrotational flows may occur in both viscous and inviscid fluids. An independent mathematical proof of potential flow as a scalable description of irrotational flow in viscous fluids (in compressible) is given in [25]. The common misperception that potential flow and stream function descriptions would only apply to inviscid fluids probably follows from an overly cautious approach: inviscid fluids will always flow in irrotational fashion and therefore automatically fulfill the requirements of potential flow. However, irrotational flow can also occur in viscous fluids. In other words, a fluid that is able to exhibit an irrotational flow is not necessarily an inviscid fluid. This subtle difference was ignored, as detailed in Joseph [28], by many authors of classical texts on fluid mechanics and potential flow theory. Joseph [28] states: “Every theorem about potential flow of perfect fluids with conservative body forces applies equally to viscous fluids in regions of irrotational flow”. A more relaxed formulation is given in Joseph et al. [29]: “Viscous potential flow is a potential flow solution of the Navier–Stokes equation in which the vorticity vanishes and no-slip conditions at interface are not enforced”. One could amplify that if solid boundaries (interfaces) are not occurring near the flow region studied, the flow description is valid for an infinite (unbounded) viscous continuum for which conditions of no-slip do not appear relevant.

Appendix B: Specific velocity potential

B.1 Stream function from complex potential

The stream function \( \psi \) can be mathematically expressed as part of a complex potential \( \Phi \) or \( W(z) \), which links the generic potential function \( \phi \) and the stream function \( \psi \):

\[
W(z) = \Phi = \phi + i\psi
\]  

(B1)

The related complex function \( F(z) \) is the conjugate of the complex potential \( W(z) \):

\[
F(z) = \overline{W(z)} = \phi(x, y) - i\psi(x, y)
\]  

(B2)

The derivative of \( \overline{F(z)} \) yields the Pólya vector field or velocity potential, which is given here in Cartesian \((x,y)\) coordinates; polar coordinate solutions also exist [33]. Using \( \nu_x = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \) and \( \nu_y = -\frac{\partial \psi}{\partial x} \) yields:

\[
V(z) = \frac{dF}{dz} = \frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x} = \nu_x - i\nu_y
\]  

(B3)

\( V(z) \) is analytic if and only if the Pólya vector field \( \overline{V(z)} \) is irrotational and incompressible, so that Cauchy-Riemann equations are fulfilled (Theorem 3.4 of Brilleslyper et al. [34]).

B.2 Complex potential and velocity potential for source flows

For example, nested source flows can be described by specifying the real and imaginary position, respectively \( x_i \) and \( y_i \), of the sources. The specific complex potential of the 2D vector field produced by \( n \) sources in the complex plane is [35]:

\[
W(z) = \sum_{s=1}^{n} m_i \log(z - z_s)
\]  

(B4)

Applying Eqs. (B1) and (B2) gives the description of the vector field for such nested source flows by a polynomial function in the form of complex series [34]:

\[
V(z) = \sum_{s=1}^{n} \frac{m_i}{(z - z_s)}
\]  

(B5)

The above velocity potential is a vector field, which is analytic as follows from the fact that the conjugate Pólya vector field \( \overline{V(z)} \) is irrotational and incompressible [33].

Streamlines in the fluid continuum can be traced when we know the velocity vectors in \( x \) and \( y \) direction \((\nu_x \text{ and } \nu_y)\) throughout the flow space and these are given by \( V(z) = \nu_x - i\nu_y \). For the multiple sources we can obtain the velocity in the \( x \) and \( y \) directions from the following derivation:

\[
V(z) = \sum_{s=1}^{n} \frac{m_i}{z - z_s} = \sum_{s=1}^{n} \frac{m_i}{(x + iy) - (x_s + iy_s)} = \sum_{s=1}^{n} \frac{m_i}{(x - x_s) + i(y - y_s)}
\]

\[
= \sum_{s=1}^{n} \frac{m_i}{(x - x_s) + i(y - y_s)} \cdot \frac{(x - x_s) - i(y - y_s)}{(x - x_s) - i(y - y_s)} = \sum_{s=1}^{n} \frac{m_i((x - x_s) - i(y - y_s))}{(x - x_s)^2 + (y - y_s)^2}
\]
\[
\begin{align*}
&= \sum_{i=1}^{n} \frac{m_i (x - x_i)}{(x - x_i)^2 + (y - y_i)^2} - i \sum_{i=1}^{n} \frac{m_i (y - y_i)}{(x - x_i)^2 + (y - y_i)^2} = v_x - iv_y.
\end{align*}
\]

Hence we find:
\[
\begin{align*}
&v_x = \sum_{i=1}^{n} \frac{m_i (x - x_i)}{(x - x_i)^2 + (y - y_i)^2} \\
&v_y = \sum_{i=1}^{n} \frac{m_i (y - y_i)}{(x - x_i)^2 + (y - y_i)^2}
\end{align*}
\]

These are the two velocity vectors \((v_x \text{ and } v_y)\) in x and y direction throughout the flow space, which thus describe the full Pólya vector field.

\subsection{B.3 Velocity gradient tensor for source flows}

Differentiation of the Pólya vector field \(v(z)\) with respect to x and y yields the velocity gradient tensor for our nested source flows:

\[
\nabla v = \begin{bmatrix}
\sum_{i=1}^{n} \frac{m_i - (x - x_i)^2 + (y - y_i)^2}{((x - x_i)^2 + (y - y_i)^2)^2} \\
\sum_{i=1}^{n} \frac{2(x - x_i) \cdot (y - y_i)}{(x - x_i)^2 + (y - y_i)^2} \\
\sum_{i=1}^{n} \frac{2(x - x_i) \cdot (y - y_i)}{(x - x_i)^2 + (y - y_i)^2} \\
\sum_{i=1}^{n} \frac{m_i - (x - x_i)^2 - (y - y_i)^2}{((x - x_i)^2 + (y - y_i)^2)^2}
\end{bmatrix}
\]

The velocity gradient tensor \(\nabla v\) can be decomposed in the symmetric (strain rate tensor) and anti-symmetric (rotation rate tensor) parts, again using \(\nabla v = \frac{1}{2}[(\nabla v + \nabla v^T)] + \frac{1}{2}[\nabla v - \nabla v^T]\), hence:

\[
\nabla v = \begin{bmatrix}
v_{xx} & \frac{1}{2}(v_{xy} + v_{yx}) \\
\frac{1}{2}(v_{yx} + v_{xy}) & v_{yy}
\end{bmatrix} + \begin{bmatrix}
0 & \frac{1}{2}(v_{xy} - v_{yx}) \\
\frac{1}{2}(v_{yx} - v_{xy}) & 0
\end{bmatrix}.
\]

Expression \([B8]\) is scaled only by the strength, \(m_i\), which specifies the 2D flow strength of each source (positive for a source and negative for a sink), which can be applied to describe flow in a horizontal reservoir of thickness, \(h [m]\), and assuming a volumetric well rate of \(Q_s [m^3]\):

\[
m_i = Q_s / 2\pi h \quad [m^2 s^{-1}]
\]

The flux \(Q_s\) of a well source is given by (for \(r > 0\)):

\[
Q_s = h \int_0^{2\pi} \nu_r r d\theta \quad [m^3 s^{-1}]
\]

The radial velocity \(v_r\) of each individual source is (for \(r > 0\)):

\[
v_r = m_i / r \quad [ms^{-1}]
\]

Note that the scaling of the strength, \(m_i\), in expression \([B10a]\) includes \(2\pi\) in the denominator. If \(m_i\) is instead scaled by \(m_i = Q_s / h\), then Eqs. \([B4]\) through \([B8]\) need to use \(m_i / 2\pi\) instead of \(m_i\) only. Examples of source flows with scaled strengths have been given in prior study for flow studies of flooding in porous media [17,18,36].

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